Royal School
of
Bridges and roadways

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Notes on applied mechanics relating to the principles of structural stability and the dynamic theory of machines.

Chap. 1. Recapitulation of the essential notions of rational mechanics

§: 1. On the various branches of mechanics.

1. Mechanics has as its object the laws and causes of motion and equilibrium of bodies.

2. The general truths, without exception, belonging to this science constitute Rational Mechanics, so named because its propositions are consequences that reasoning rigorously deduces from a small number of principles or laws of nature.

However, Rational Mechanics admits, in several of its theories, the existence of bodies possessing properties, such as perfect hardness, perfect fluidity, etc., that real bodies never completely possess, and the propositions that science arrives at with the help of these hypotheses are only verified with more or less approximation in the real world.

3. Rational Mechanics is divided into two parts: statics or the science of equilibrium, and dynamics, which deals with the relations of motion with its causes, called forces.

4. Rational mechanics is the necessary basis of various sciences which are like branches of Mechanics taken in its most general sense: such are mathematical Astronomy or Celestial Mechanics, mathematical Physics, Hydraulics, the Theory of structural stability, the science of Machines or Industrial Mechanics.

The last three are essential parts of the knowledge useful to Bridge and Road Engineers.

The science of Machines, or Industrial Mechanics, can be divided into several very important and so extensive parts that it cannot be given to anyone to possess them all completely.

The first, which principally considers in machines the transmissions of movement, and the relations to which the various movements of their parts are subject, without dealing with calculating the intensity of the motor and resistances, could be called Geometrical Mechanics

because the very varied and often very ingenious processes it employs are almost exclusively founded on the rules of geometry.

7. The third part relates to the construction of machines: it includes the knowledge of materials, the art of assembling them, and the rules proper for determining the shape and dimensions of each piece according to the forces it receives.

8. The fourth part of Industrial Mechanics, which can be designated by the name of Mechanical Technology, relates to instruments, called machine tools, which, driven by appropriate movement transmission devices in each case, immediately execute the various operations of industry. The objects with which this part of Mechanics deals are in reality the goal towards which the other three tend, but science possesses with regard to them few general precepts, and they belong for the most part to descriptive teaching and to practice.

9. Having taken last year as the subject of our lessons general hydraulics and hydraulic machines, we will deal this year with the other parts of Mechanics which interest Engineers. The subject matter is so extensive that we will often be obliged to limit ourselves to the most important generalities. Moreover, we will leave to the special construction courses the developments that belong to them. Before all else, we will present a summary of the essential notions of Rational Mechanics which will serve us in the rest of this course, and we will fix the meaning of some expressions we will have to use.

§: 2. On motion considered independently of its causes.

10. The motion of a point, if the straight or curved line it traverses is known, is determined as soon as, having chosen a fixed point on this line, and the positive direction in which it is conceived to be generated, we have the relation between the positive or negative distance of the moving point to this fixed point, and the time elapsed from an instant called the initial. From this relation, which can be expressed either analytically by an equation , or gr $s = F(t)$ aphically by a curve, we deduce the velocity at a given instant, whose analytical expression is , $\gamma = \frac{ds}{dt}$ and whose value, positive or negative, can be obtained either by calculating this derivative, or by drawing the tangent to the representative curve of the motion.

11. When the velocity is constant, the distance s varies proportionally to time. The movement is then said to be uniform; its analytical expression is or $\gamma = a$. $s = s_0 + at$ The velocity a is moreover positive or negative. For a given uniform motion, the numerical expression of a depends on the choice of units of length and time. In practice, it is only in uniform motion that the velocity of a point can be directly observed.(*)

(*) See the memoir by Mr. Morin entitled *Description of chronometric devices with stylus suitable for the graphical representation and determination of the laws of motion (Description des appareils chronométriques à style propres à la représentation graphique and à la détermination des lois du mouvement)*, 1838.

12. When the velocity of a point is variable, one is led to consider a quantity which is the acceleration; its analytical expression is $\frac{dv}{dt}$, and its positive or negative value at a given instant, is obtained either by calculating this derivative, or graphically by drawing the tangent to the corresponding point of a curve whose coordinates are proportional to time and to velocity. When the velocity and the acceleration have the same sign the movement is accelerated, in the vulgar sense of this word; in the opposite case it is retarded.

When the acceleration is constant, (positive or negative), the movement is said to be uniformly varied; and we have $\frac{dv}{dt} == j$ hence $\gamma = \gamma_0 + jt$, hence $s = s_0 + \gamma_0 t + \frac{1}{2} jt^2$.

Any one of these three equations implicitly contains the other two, except for the determination of the constants s_0 and γ_0 .

13. When the line that a moving point describes is not known *a priori*, its movement is determined as soon as we know the movement of its projections onto three coordinate axes. The velocity of the point in space is then essentially positive and represented, in magnitude and in direction, by the diagonal of the parallelepiped whose three contiguous sides represent in magnitude and direction the positive or negative velocities of the projections of the moving point onto the coordinate axes.

14. A point M having any movement whatever, if one conceives that three coordinate planes forming a trihedral angle of invariable shape also move in some way in space, and if one supposes that an observer, unconsciously carried along in the movement of these axes, considers them as fixed, we see that this observer will attribute to point M a movement different from that which this moving point really possesses: this apparent movement is called relative movement in the system of movable axes supposed fixed. The velocity of this movement is called relative velocity.

15. If at a given instant we consider the position A of the moving point M, this geometric point A, considered as invariably linked to the movable axes, possesses a movement depending uniquely on that of these axes. This is what Mr. Coriolis called the transport motion of the moving point M at the instant considered.

The current velocity of point A is the transport velocity of point M.

16. It is easily demonstrated that the absolute velocity is represented in magnitude and in direction by the diagonal of a parallelogram whose two contiguous sides likewise represent, one the transport velocity, the other the relative velocity.

17. The various movements of a solid system can be classified as follows.

1° Common translational motion, curvilinear or rectilinear, which occurs when all points of the system possess at each instant equal, parallel velocities in the same direction.

2° Simple motion about a fixed axis. In this case, frequent in machines, the velocities of the various points of the system, at the same instant, are proportional to the distances r of these points from the fixed axis of rotation. This is expressed by the formula $\gamma = wr$ in which the quantity ω, the velocity of the point which, linked to the system,

would be at unit distance from the axis, is called the angular velocity of the solid system at the instant considered. If ω is variable with time, the quantity $\frac{dw}{dt}$ is the angular acceleration, which is constant when the rotational movement is uniformly varied.

3° Cylindrical rolling motion, occurring when the solid system is invariably attached to a cylindrical surface, with any base, which rolls without slipping on another fixed cylindrical surface. At any instant, the directions and ratios of the velocities of the different points of the system are the same as if, at that instant, the contact edge of the two cylinders were fixed. This edge, which is not really fixed but whose velocity is zero at the instant of contact, is called the instantaneous axis of rotation. In reality, the arcs described, even during very short times, are not arcs of circles.

4° Rotation about a fixed point which occurs when the system is invariably linked to a point which remains fixed during the movement of the system. At any given instant, there is an instantaneous axis of rotation as if the system were linked to a conical surface rolling without slipping on another which would be fixed and have the same vertex.

5° Movement composed of translation and rotation about an axis, occurring when a straight line *AB*, invariably linked to a solid system, has a common curvilinear or rectilinear translational movement, and consequently remains parallel to its initial position, while the points of the system situated outside this straight line have a different movement. It is then said that the movement of the solid system is composed of the translational movement of the straight line *AB* and the relative rotational movement about this axis, an expression which relates to the notions expressed in Nos. 14, 15, 16.

6° Movement composed of translation and rotation about a point. Whatever the movement of a solid system, if one imagines that through one of its points *A*, one passes three coordinate axes carried along with this point A but remaining parallel to their initial directions, the movement of the system relative to these three axes reduces to a rotational movement about point *A*, considered as fixed (4°). Thus the most general movement of a solid system can be considered as composed of the translational movement determined by the absolute movement of any one of its points and of a relative rotational movement about this point.

§: 3 Of forces considered independently of the measurement of their effects.

18. A force is the necessary and sufficient cause to modify the magnitude or the direction of the velocity of a material point. We acquire the idea of it through the sensation of the efforts we exert on bodies.

The point upon which a force acts, and without which it cannot exist, is called a point of application.

Several forces can be applied simultaneously to the same point along the same straight line.

The single force to which they are equivalent is called their sum. Hence the notion of the ratio of two forces.

Every force is comparable to a weight, i.e., to the action of gravity on a certain body, in a determined place. The unit of force is arbitrary. In numerical applications we will adopt the Kilogram.

19. If we project a force F on an *x-axis*, we have a force $F\cos(F\ ,\ x)$ or, for short, F_x, positive or negative, while the force F in space usually has no sign.

20. It is generally agreed today to no longer admit instantaneous forces supposed to have no duration and capable of producing in bodies abrupt changes in velocity, without making them pass through intermediate states. Every force therefore has a certain duration.(*)

21. If the projection F_x of a force is constant during a time t, the product $F_x t$ which appears in this important statement will be called the impulse of the force F along the *x-axis* during the time t. If the force F_x is variable, its impulse during the time $t - t_0$ is:

$$\int_{t_0}^{t} F_x\, dt\ .$$

22. The point of application of the force F describing a straight or curved line, from any causes, if we denote by ds an element of the path traveled, the quantity $F\ ds\cos(F\ ,\ ds)$ is the elementary work of the force F during the traversal of the arc ds. We can consider it under two aspects: $F\ ds\cos(ds\ ,\ F)$ and $ds\ x\ F\cos(F\ ,\ ds)$.

The integral of this differential is the work of the force F between the same limits. This is expressed by the notation $\varphi F = \int F\ ds\cos(F,\ ds)$. . According to whether it is positive or negative, the work is called motive work or resistant work.

The force whose work is positive or motive is a moving force; that whose work is negative or resistant is a resisting force.(*)

23. The expression work of a force, proposed by Mr. Coriolis and adopted by Mr. Poncelet in his courses, is not yet exclusively used by Mechanicians. This same quantity has been called by Monge and Hachette dynamic effect, by Coulomb and Navier quantity of action. We will adopt as unit the Kilogram-meter = 1kg × 1m. Montgolfier, Hachette, Clément have called a work of 1000 Kilogram-meters a dynamic unit or dynamic; Mr. Coriolis has called it a dynamode. Mr.

(*) The doctrine which repels instantaneous forces was professed by *M*. Poncelet at the School of Artillery and Engineering of Metz, as early as 1825 and by *M*. Coriolis in his *treatise on the calculation of the effect of Machines* published only in 1829, but written several years before. It was admitted by *M*. Poisson in the 2nd edition of his treatise on Mechanics in 1835.

(*) See the thesis already cited by Mr. Morin for the description of the devices suitable for measuring either the pulse or the work developed by the motor.

Dupin has called a dyname a continuous work which would amount to one million kilogram-meters in 24 hours: usage has not consecrated these denominations so far.

24. The work of a force for a determined path of its point of application is, by its definition, a quantity independent of time, i.e., of the duration of this path. One sometimes needs to designate an indefinitely prolonged work, which maintains equal values during equal times; one could take as a unit for this work a Kilogram-meter per second, but practical mechanicians are beginning to agree to call a steam-horse or dynamic horse a continuous work of 75 kilogram-meters per second. We think it is improperly that this quantity is sometimes named horse-force, an incorrect translation of the English expression horse power, power of horse (*puissance de cheval*), in which the word power has the sense of work capacity. It is important not to confuse the simple idea of force with the complex idea of work. A force is expressed in Kilograms; a work or a mechanical power is expressed in Kilogram-meters.

25. In calculating the effect of forces applied to the various points of a material system, one often has to consider the algebraic sum of the works of these forces (39), hence the usefulness of the following propositions, the second of which is moreover evident.

26. Theorem. When two equal and directly opposite forces are applied to two points in motion, if the intensity of these forces is constant or varies only as a function of the distance of the two points, the sum of their works depends only on the relative motion of the two points, and is consequently equal to the work that one of the two forces would produce alone, if the motion of the point to which it is applied were reduced to moving it away from or bringing it closer to the other point supposed fixed.

27. Theorem. If a solid system possesses a common translational motion (17, 1°), the sum of the elementary works of the forces applied to it, during an infinitesimal displacement of the system, is equal to the elementary path of one of its points multiplied by the sum of the projections of the forces onto an axis parallel to the translational velocity.

28. Theorem. If a solid system possesses a rotational motion about a fixed axis, the sum of the elementary works of the forces applied to it, during an infinitesimal angular displacement of the system, is equal to this angular displacement (i.e., to the ratio of the arc described by any point to its radius) multiplied by the algebraic sum of the moments of the forces with respect to the axis of rotation.

By moment of a force with respect to its axis is meant here the product obtained by multiplying the projection of the force onto a plane perpendicular to the axis by the distance of the force from the axis, and affecting the result with the sign + or -, which is appropriate for the work of the force in the rotation in question. Considering a force F, its moment with respect to an *x-axis* will be denoted by the notation

$$M\{F\}_x .$$

§: 4. Dynamics of a material point.

1° Rectilinear motion

29. If the force acting on a material point moving on a vertical line is its weight . . . . . ***P***.

Its acceleration at any instant is independent of the nature of the body, and of its acquired velocity. In Paris, when taking the meter and the second as units, this acceleration is 9.8088: we will always denote it by . . . . . **g**.

Under another force . . . . . ***F***.

Directed along the same straight line as the previously acquired velocity, the same body takes an acceleration $\frac{dv}{dt}$ denoted by . . . . . **J**.

That established, a law of nature which can be called the principle of relative translational motions gives

$$\frac{F}{J} = \frac{P}{g}$$

30. If two bodies under the same force take the same acceleration they are said to have the same mass. If not, their masses are proportional to the values of the ratio $\frac{F}{J}$, variable for diverse bodies, but constant for the same body; and as it is agreed to take as unit of mass that of the body which under the unit of force takes the unit of acceleration, the ratio $\frac{F}{J}$ becomes the numerical expression of the mass: if we represent it by m, we have:

$$\frac{F}{J} = m \quad \text{or} \quad \frac{F}{m} = \frac{dv}{dt} \quad \text{or} \quad F\ dt = m dv\ ;$$

taking the positive direction of the force and that of the velocity in the same sense, and giving to F and dv the signs which belong to them as a result of this convention.

31. From the last equation we obtain

$$\int_{t_0}^{t} F\ dt = mv - mv_0$$

i.e., that the impulse of the force F is equal to the algebraic increase in the quantity of motion between the same limits.

32. The two equations

$$F\ dt = m dv \qquad \text{and} \qquad \frac{dx}{dt} = v$$

allow elimination of dt and give

$$F\ dx = mv dv \quad \text{hence} \qquad \int_{x_0}^{x} F\ dx = \frac{1}{2} mv^2 - \frac{1}{2} mv_0^2$$

$\int F\ dx$ is the work of the force F. If $v_0 = 0$, the quantity $\frac{1}{2} mv^2$ is equal to the positive or motive work which imparted this velocity to the body of mass m; it therefore represents the

effect of this work. If $v = 0$ the quantity $-\frac{1}{2} mv_0^2$ is equal to the negative or resistant work that the body has received from the instant when its velocity was v_0 until that when this velocity was annihilated; the quantity $\frac{1}{2} mv_0^2$ therefore expresses the capacity that the considered body possessed to overcome a resistant work by virtue of its mass and its velocity.

This quantity is half of what most authors call living force, an expression born from the application of the word force to the work capacity of motors. We call it living power, an expression in which the word power corresponds to the idea of this capacity, and the epithet living corresponds to that of velocity.

2° Curvilinear motion

33. The preceding notions extend to curvilinear motion, which can be considered in two ways.

This motion can be projected onto an axis, and the equations of Nos. 30, 31 and 32 subsist provided we replace 1° the force F by the sum $\sum F_x$ of the projections of forces onto the axis, supposed to be the *x-axis*; 2° the velocities v, v_0 by their projections v_x, v_{0x} onto the same axis.

This is expressed by saying that: the projection of a material point onto an axis moves like a material point of the same mass, which would have as initial velocity the projection of the initial velocity of the point in space, and which would be acted upon by a force equal to the algebraic sum of the projections of the simultaneous forces.

It is easily demonstrated that, whatever the forces applied to the same point, one can find a single force, called their resultant, whose projection onto any axis would be equal to the sum of the projections of the actual forces considered.

Denoting this resultant by R, the analytical expression of the preceding statement is

$$\frac{dv_x}{dt} = \frac{R_x}{m} \quad \text{hence} \qquad mv_x - m_{0x} = \int R_x \, dt \ ,$$

formulas in which R_x can be replaced by $\sum F_x$.

Curvilinear motion can be considered along the curve actually described, but abstraction made of the curvature of the traveled path. The equations of Nos. 30, 31 and 32 still exist, provided we replace 1) the rectilinear distances x, x_0 by the distances s, s_0 measured along the described curve, and 2° the force F by the tangential force $\varphi = \sum F \cos(F\ ,\ d_s)$ or $R \cos(R\ ,\ d_s)$.

We thus have

$$v = \frac{ds}{dt} \ , \quad \frac{dv}{dt} = \frac{\sum F \cos(F\ ,\ ds)}{m} = \frac{R \cos(R\ ,\ ds)}{m}$$

hence

$$mv - mv_0 = \sum \int F \cos(F\ ,\ ds)\, dl = \int R \cos(R\ ,\ ds)\, dt$$

and

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \sum \int F \cos(F\ ,\ ds) = \sum \zeta\{F\} = \zeta\{R\}\ .$$

If we decompose the resultant R into two rectangular forces, one of which is the tangential force φ, the other component ψ lies in the plane of the described curve, or in the osculating plane, if the curve has double curvature, along the normal to this curve, and in the direction going from the curve to the center of curvature. This normal force is the centripetal force; its expression is, denoting by r the radius of curvature

$$\Psi = \frac{mv^2}{r} \qquad \text{or} \qquad \psi = mw^2\, r$$

calling ω the angular velocity about the center of curvature.

§: 5 General principles of the motion of a system of material points

A system of material points is subjected to two kinds of forces, some external, due to bodies not forming part of the system, the others mutual, between the elements of the system.

The theorem of the quantity of motion of any system can be reduced to these terms: the sum of the quantities of motion projected onto any axis is independent of mutual actions, and consequently, its increase is equal to the sum of the impulses of external forces projected onto the same axis. The analytical expression of this principle is, denoting by F the external forces,

$$\sum mv_x - \sum mv_{0x} = \sum \int F_x\, dt\ .$$

The theorem of the motion of the center of gravity consists in that the center of gravity of any system moves like a material point which would alone unite the mass $\sum m$ of the system, and which would be acted upon by the external forces transported parallel to themselves to this point. Thus all the properties recalled in §: 4 are applicable without any modification to the motion of the center of gravity of any material system, solid or not solid, by means of the fictitious transport of the external forces and the mass of the system to this point.

In the case where the body is solid, not only does the center of gravity move as if all the forces were transported there, but in addition, the body rotates about this point as if it were fixed, the external forces remaining the same in intensity and direction, passing through their actual points of application.

39. The theorem of the effect of work consists in that the increase in the living power of the system is equal to the work of external forces, plus that of mutual actions, depending uniquely on the relative motion of the points of the system.

The analytical expression of this principle is

$$\sum \frac{1}{2}mv_x - \sum \frac{1}{2}mv_0^2 = \sum \zeta\{F\} + \sum \int fdl$$

denoting by f the mutual forces, supposed positive when they are repulsive, and by dl the variation in distance between the two points of the system between which the force f is exerted.

40. In the case where the system is supposed solid, the distance variations dl are zero, and the term $\Sigma \int fdl$ disappears. The same disappearance is admissible in the case of a slightly viscous liquid whose molecules have a relatively slow motion. Experience teaches in which case this hypothesis is permissible.

41. The term $\Sigma \zeta F$ takes a simple expression when it is a question of the work of gravity: it is equal to the weight of the system, multiplied by the vertical height through which the center of gravity has descended. If the center of gravity had risen, this work would be negative.

42. The term $\Sigma \frac{1}{2} mw^2$ also takes a simple expression when it concerns a solid body rotating about an axis. This expression of living power is:

$$\frac{1}{2} w^2 \sum mr^2$$

ω being the angular velocity, m the mass of one of the points of the system and r its distance from the axis of rotation. The sum Σmr^2 is called the moment of inertia of the solid body with respect to the fixed axis about which it rotates.

43. The calculation of the work of forces is the most essential object of the dynamic theory of machines. One deals from a secondary point of view with the intensities of the forces which act on the various points of these apparatus to ensure the solidity of their construction and to be able to take friction into account. Thus the effort or pressure that a motive agent is capable of exerting on its receiver, i.e., on its points of application in a machine, is of accessory importance; it is the same with the intensity of the force that must be exerted at certain points of a machine tool to start it or to maintain it in motion. The principal question, as regards the motor, is to know what is the work it exerts on its receiver, and, as regards the machine tool, what is the work it requires to function.

These two questions being resolved, by the combination of experience and theory, the general principle of the effect of work shows that if it were possible to establish between the receiver and the tool a transmission device without shocks or friction, it would suffice to divide the work that the motor gives, by that which a machine-tool requires, to know how many similar machines the motor would keep in motion. This result is an upper limit that can never be attained because from the motive work must always be subtracted the resistant work due to friction and shocks of the parts of the transmission device among themselves, and against their supports.

44. Much dispute arose in the 18th century over the measurement of the forces of bodies in motion. In the very simple case of a solid body possessing a common translational motion, the question was reduced to knowing "whether the force of a body which has a certain velocity, becomes double or quadruple when its velocity becomes double"(*). It is evident that the word

(*) *D'*Alembert: *Eulogy of Bernoulli.*

force was then given a meaning different from that which we adopt (18); and in fact one admitted the existence of instantaneous forces (20), and in speaking of a body in motion, and abandoned to itself, one said that its force remained the same, that this force was constantly preserved,(**) as if it were at each instant the actual cause of the motion. According to modern ideas, force is not the subsisting cause of any existing motion, but the cause which modifies any variable motion. A body abandoned to itself neither exerts nor receives actually any force; and if one asks what is the force which produced this movement or what is that which would annihilate it, this question is indeterminate: but the quantity of motion mv of the body considered makes known the impulse $\int F\,dt$ of the force in question, which is double when the velocity becomes double; while the living power $\frac{1}{2}mv^2$ makes known the work $\int F\,dv$ of this force, which becomes quadruple when the velocity is double. Thus the philosophers who like Newton and Euler, measured what they called the force of a body in motion by the product of its mass by its velocity, called force the complex quantity that we call impulse; and those who like Leibniz and Bernoulli took as measure the product of the mass by the square of the velocity gave the name force to the complex quantity that we name work.

§: 6 Statics or general conditions of the equilibrium of bodies

45. When a material system, solid or not solid, is at rest, each of its points is in equilibrium with the influence of the forces F proper to it, and the mutual actions f it receives from the other points of the system.

If one imagines that under the action of the forces F the system is otherwise free in space, one can conceive an infinity of possible movements of the system in which the forces accompany their points of application. These movements which exist only in thought are called virtual movements and are ordinarily supposed infinitesimal.

If, among the virtual movements, we consider only those in which the respective distances of the points of the system remain invariable, we have the virtual movements compatible with the solidity of the system.

The system being in equilibrium, it is evident that to produce actually one of these latter movements it would suffice to apply simultaneously to each of the points of the system a certain force φ, constant or variable, capable of determining the movement proper to that point, as if it were free. Therefore, applying to each point the theorem of No. 32, and summing the equations thus obtained, we have, after any displacement:

$$\frac{1}{2}\sum mv^2 = \sum \zeta\{\varphi\}$$

but since it concerns a system then subjected to the two kinds of external forces designated by F and φ, and since the mutual actions produce no work in this system whose various points maintain their distances, we have according to No. 39,

$$\frac{1}{2}\sum mv^2 = \sum \zeta\{F\} + \sum \zeta\{\varphi\}.$$

(**) Laplace: *Exposition of the system of the world*. Book 3 Chap. 2

From these two equations we conclude $\sum \zeta \{F\} = 0$; this constitutes the theorem of virtual work (or, following the ordinary expression, the theorem of virtual velocities) which can be stated thus:

Theorem. A material system being in equilibrium, the algebraic sum of the work of external forces is zero for any virtual movement compatible with the solidity of the system.

From this are deduced, as will be seen, the six general conditions of the equilibrium of a system.

46. If for virtual movement one chooses a common translational movement parallel to an *x-axis*, calling ξ the common displacement of all the points we will have (27)

$$\sum \zeta \{F\} = \xi \sum F_x \ ; \qquad \text{therefore} \qquad \sum F_x = 0.$$

Thus for three coordinate axes we have

$$\sum F_x = 0 \ , \qquad \sum F_y = 0 \ , \qquad \sum F_z = 0 \ ;$$

which amounts to saying that the center of gravity of the system remains at rest, if it is there initially.

47. If for virtual movement one chooses a rotational movement about an *x-axis*, denoting by σ the arc described by a point linked to the system and situated at unit distance from the axis, we have (28)

$$\sum \zeta \{F\} = \sigma \sum M\{F\}_x \ ; \quad \text{therefore} \ \sum M\{F\}_x = 0 \ .$$

Thus for three coordinate axes we have

$$\sum M\{F\}_x = 0 \ , \qquad \sum M\{F\}_y = 0 \ , \qquad \sum M\{F\}_z = 0 \ .$$

48. When the system considered is supposed solid, an hypothesis which is never perfectly realized, the six preceding conditions are, moreover, sufficient for equilibrium.

To demonstrate this we first establish the following proposition:

Theorem. When in a system of points of application of any forces one considers only virtual movements compatible with solidity, the forces whatever they may be can always be reduced to two, one of which at least passes through a given point, without changing anything in the sum of the virtual works; i.e., that in all movements of the indicated kind these two forces, said to be equivalent to the real forces, produce the same sum of virtual work as would the real forces.

When the two equivalents reduce to a single one, this single force is called the resultant of the real forces.

49. That established, it is easily demonstrated that when the six equations of Nos. 46 and 47 hold, the two equivalents are equal and opposite; that thus the resultant is zero; that consequently the system, if supposed solid, is necessarily in equilibrium, because there cannot be acquired velocity without work.

This proposition and those of Nos. 46 and 47 are summarized in these terms:

Theorem. In a system in equilibrium the sum of the projections of external forces, onto any axis, is zero, and the sum of the moments of these same forces with respect to any axis is

likewise zero. If the system is solid, for equilibrium to occur, it suffices that the double condition of projections and moments holds for three concurrent axes not lying in the same plane.

50. The particular cases of the equilibrium of forces situated in a plane, and of parallel forces in space, are included in this general theorem.

Chap. 2. Application of statics to articulated systems without friction.

§:1 *Theory applicable to suspension bridges*

51. The total load is supposed to be distributed at the attachment points of the vertical suspension rods; thus the chain or cable is thereby assimilated to a funicular polygon. The tensions of the rods are supposed to be the same as if the spans, instead of forming a rigid system, were simply articulated, so that the equidistant rods support equal loads.

Here is a theorem on this subject which does not assume equidistant rods; ; the following demonstration was indicated by *M*. Guillebon (*Annales des Ponts and chaussées* - January 1839):

If the vertices of a funicular polygon support vertical loads proportional to the distances between the suspension rods, i.e., such that δ', δ'' being the horizontally projected distances of three consecutive vertices, the load on the intermediate vertex is represented by $p\frac{\delta'+\delta''}{2}$, denoting by p the weight corresponding to a unit of horizontal length, all the vertices lie on a parabola of the second degree whose principal axis is vertical.

Let (fig. 1)

$M_0, M_1, M_2, \ldots\ldots\ldots M_n,$ be several consecutive vertices of the funicular polygon, comprising between them n sides of this polygon,

$Y_0, Y_1, Y_2, \ldots\ldots\ldots Y_n,$ their vertical ordinates above any horizontal line 0_x ,

$X_0, X_1, X_2, \ldots\ldots\ldots X_n,$ their abscissae $OP_0, OP_1, OP_2, \ldots\ldots\ldots OP_n,$ from O, the projection of the midpoint of the side NM_0 which precedes the vertex M_0 ,

$\alpha_0, \alpha_1, \alpha_2, \ldots\ldots \alpha_n,$ the angles that the sides $NM_0, M_0M_1, M_1M_2, \ldots\ldots\ldots M_{n-1}M_n$ make above the horizon.

$T_0, T_1, T_2, \ldots\ldots\ldots T_n,$ the tensions of these same sides.

The portion $M_0, M_1, \ldots\ldots\ldots M_{n-1}$ of the polygon is subjected to two kinds of external forces; some are vertical, and their sum is equal to p multiplied by $\frac{x_n+x_{n-1}}{2}$, the distance from point O to the midpoint of the interval $P_{n-1}P_n$; the others are the two tensions T_0, T_n directed the first along $M_0N,$, the second along $M_{n-1}M_n$

Applying to these forces the formulas of No. 46, we have

$$T_n \sin \alpha_n = p\,\frac{x_n + x_{n-1}}{2} + T_0 \sin \alpha_0 \ ,$$

and $$T_n \cos \alpha_n = T_0 \cos \alpha_0$$

This last equation, applicable to any two tensions, expresses that the horizontal projections of all the tensions have a constant value. We will represent it by T_x .

The two preceding equations give, according to this notation

$$\tan \alpha_n = \frac{p}{2\,T_x}(x_n + x_{n-1}) + \tan \alpha_0$$

but $$\tan \alpha_n = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} .$$

Substituting we find:

$$y_n - y_{n-1} = \frac{p}{2\,T_x}(x_n^2 - x_{n-1}^2) + \tan \alpha_0 (x_n - x_{n-1})$$

if in this formula we successively put $n = 1, 2, 3 \ldots\ldots\ldots n$ we obtain n equations.

Adding them and then removing the index n we obtain the formula applicable to any sum

$$y - y_0 = \frac{p}{2\,T_x}(x^2 - x_0^2) + \tan \alpha_0 (x - x_0) ,$$

which establishes the stated proposition.

Remark. If we change the origin of coordinates, transporting the axes parallelly, the equation will retain the same form

$$y = \alpha\, x^2 + bx + c ,$$

and the coefficient α of x^2 will always be $\frac{p}{2\,T_x}$.

52. Three points, or two points and the situation of the horizontal tangent, suffice to determine the parabola in question.

The coefficient α of x^2 being found as a function of the coordinates of the given points, we deduce from it T_x by the relation $\alpha = \frac{p}{2\,T_x}$

hence $$T_x = \frac{p}{2\,\alpha} .$$

We will then obtain any tension T_n by the formula $T_n \cos \alpha_n = T_x$ in which we set

$$\cos \alpha_n = \frac{x_n - x_{n-1}}{\sqrt{(y_n - y_{n-1})^2 + (x_n - x^{n-1})^2}}$$

53. The application of the preceding theory to one of the chains of a suspension bridge is immediate when the weights to be supported are supposed proportional to the lengths of the parts of the floor projected horizontally. It must be observed that point A' (fig. 2), where the chain meets the vertical line from the floor support point on the abutment, is on the parabola which continues all the vertices of the polygon, but that the attachment point of the chain is outside this curve if it differs from A'.

54. In practice the suspension rods are equidistant and have equal tensions. The rods situated in the same plane are often supported by several independent chains. The curve on which the vertices of each funicular polygon must then lie is the same for all the chains and

can be determined, as well as the sum of their tensions, by reasoning as if there were only a single chain.

The rods are then arranged so as to realize one of the following cases:

1st case (fig. 3) The two lower sides are symmetrical with respect to the vertical, which occurs when between two points situated at the same level there is an even number of sides.

In this case, M_0 is the vertex of the parabola. Its equation, referred to the coordinate axes O_x and O_y, is of the form

$$y = ax^2$$

The coefficient a is always linked to the horizontal tension (51) by the relation a = p/(2Tx), and the situation of a single point of the curve with respect to the axes suffices to determine the constant a, independently of the weight p and the tension T_x.. Suppose, for example, the sag $BM_0 = p$ and the half-span $A'B = c$, we will have

$$p = ac^2 \qquad \text{hence} \qquad a = \frac{p}{c^2}$$

and any vertex coordinate $$y = \frac{p}{c^2}\, x^2$$

2nd case (fig. 4) One side is horizontal; it is then the lowest of all. This occurs when between two points situated at the same level there is an odd number of sides. In this case, the vertex of the parabola is on the vertical line passing through the midpoint of the horizontal side, and, if we take as origin of coordinates this midpoint which is not on the curve, the equation is of the form $y = ax^2 + b$; and as the curve passes through M_0 we have $O = ax_0^2 + b$ and the equation becomes, (as would be obtained by setting $y_0 = 0$ and $\tan \alpha_0 = 0$ in the final equation of No. 51)

$$y = a\left(x^2 - x_0^2\right)$$

x_0 is then half the horizontal side, i.e., half the interval of any two suspension rods.

Given the sag $BO = b$ and the half-span $A'B = c$;

we will have $$p = a\left(c^2 - x_0^2\right) \qquad \text{hence} \qquad \alpha = \frac{p}{c^2 - x_0^2}$$

and for any vertex $$y = \frac{p}{c^2 - x_0^2}\left(x^2 - x_0^2\right)$$

55. In the two preceding cases, we will find the horizontal tension T_x by the relation $T_x = \frac{pc^2}{2p}$; in the second $T_x = \frac{P\left(c^2 - x_0^2\right)}{2p}$.

The tension of any side of the polygon will be found as seen in No. 52, noting that the difference $x_n - x_{n-1}$ is the constant interval between two consecutive rods; and

that $y_n - y_{n-1}$, the difference of two consecutive ordinates, will be deduced from the equation of the curve.

56. If the floor is in a plane, the length of the suspension rods being the equidistant ordinates of a parabola parallel to the principal axis, their second differences are equal among themselves. It therefore suffices to calculate three of these ordinates to obtain the others by simple additions.

The same will be true when the longitudinal profile of the floor is a parabola with a vertical principal axis.

57. It follows from the equation

$$T_n \cos \alpha_n = T_x \qquad \text{hence} \qquad T_n = \frac{T_x}{\cos \alpha_n}$$

that the tension increases with the inclination of the sides of the chain.

It increases proportionally to their length; for, calling δ the interval between two consecutive rods, and ln the length of the side whose tension is T_n, we have

$$\cos \alpha_n = \frac{\delta}{l_n}, \qquad \text{hence} \qquad T_n = \frac{T_x l_n}{\delta}$$

If α is the angle of the last side with the horizon, and T its tension, which is the greatest in the chain, we have

$$T = \frac{T_x}{\cos \alpha}$$

but this maximum tension is approximately determined by noting that the inclination of the last side is slightly less than that of the tangent to the parabola at A'. The equation of this parabola being, according to the case,

$$y = ax^2, \qquad \text{or} \qquad y = a\left(x^2 - x_0^2\right),$$

we always have $\frac{dy}{dx} = 2\,ax$. If therefore we denote by β the angle with the horizon of the tangent at the extreme point which has abscissa $x = c$, we will have

$$\tan \beta = 2\,ac \qquad \text{hence} \qquad \cos \beta = \frac{1}{\sqrt{1 + \tan^2 \beta}} = \frac{1}{\sqrt{1 + 4\,a^2 c^2}}$$

now, from $\beta > \alpha$ we deduce that $\cos \beta < \cos \alpha$, since the angles α and β are less than a right angle:

$$\text{hence} \qquad \frac{T_x}{\cos \alpha} \qquad \text{or} \qquad T < \frac{T_x}{\cos \beta},$$

or replacing T_x and $\cos \beta$ by their values,

$$T < \frac{p}{2\,a} \sqrt{1 + 4\,a^2 c^2}$$

substituting for a its value we will have in the first case, (where $a = \frac{p}{c^2}$)

$$T < \frac{pc^2}{2p}\sqrt{1+\frac{4p^2}{c^2}}$$

and in the second case (where $a = \frac{p}{c^2 - x_0^2}$)

$$T < \frac{p(c^2 - x_0^2)}{2p}\sqrt{1+\frac{4b^2c^2}{(c^2 - x_0^2)^2}}$$

When the number of suspension rods is very large, x_0 is very small compared to c, and we can, without possible error, take in both cases

$$T = \frac{pc^2}{2p}\sqrt{1+\frac{4p^2}{c^2}}$$

58. We can immediately calculate the sum of the portions of rods above the horizontal line passing through the lowest vertices of the polygon

1st case (54) The equation of the parabola is

$$y = ax^2 \qquad \text{and we have} \qquad a = \frac{p}{c^2}$$

Denote by n the number of suspension rods from the lowest point of the chain up to the vertex M_n, and by δ the constant interval between two consecutive rods, we will have

for the points	$M_0, M_1, M_2, M_3 \ldots\ldots\ M_n$
having abscissae	$0, \delta, 2\delta, 3\delta \ldots\ldots\ldots\ n\delta,$
the ordinates	$0, a\delta^2, a(2\delta)^2, a(3\delta)^2, \ldots\ a(n\delta)^2,$
whose sum is	$a\delta^2(1^2+2^2+3^2+\ldots\ldots\ldots\ +n^2).$

Now the sum of the squares of consecutive integers from *1* to n has the expression

$$\frac{1}{6}n(n+1)(2n+1)$$

therefore the desired sum is

$$\frac{1}{6}n(n+1)(2n+1)\frac{p\delta^2}{c^2}$$

2nd case (54) The equation of the parabola is in the 2nd case

$$y = a(x^2 - x_0^2)$$

where $a = \frac{p}{c^2 - x_0^2}$ and $x_0 = \frac{1}{2}\delta$

for the points	$M_0,\ M_1, M_2, M_3 \ldots\ldots\ldots\ M_n$
having abscissae	$x_0, 3x_0, 5x_0, 7x_0, \ldots\ldots\ldots\ldots\ (2n+1)x_0$
the ordinates are	$0, a(3^2-1)x_0^2, a(5^2-1)x_0^2, a(7^2-1)x_0^2, \ldots\ a[(2n+1)^2-1]x_0^2$
and their sum is	$a\,x_0^2[1 + 3^2 + 5^2 + 7^{2\ldots} + (2n+1)^2] - a\,x_0^2(n+1).$

now the sum of the squares of consecutive odd numbers has the expression

$$\frac{1}{6}(2n+1)(2n+2)(2n+3)$$

therefore, putting for x_0 and a their values $\frac{\delta}{2}$ and $\frac{4p}{4c^2-\delta^2}$, the desired sum will be equal to

$$\left[\frac{1}{6}(2n+1)(2n+2)(2n+3)-(n-1)\right].\frac{p\,\delta^2}{4c^2-\delta^2}\ .$$

59. The length l of each side of the chain can be calculated exactly by the formula

$$l_n=\sqrt{\delta^2+(y_n-y_{n-1})^2}$$

But it is useful to have immediately the total length of a half-chain, from the lowest point, or from the midpoint of the horizontal side: it can be obtained approximately.

This length differs little from that of the circumscribed parabola. An element ds of this curve has the expression:

$$\sqrt{dx^2+dy^2}\qquad\text{or}\qquad dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

now from the equations $y=ax^2$ and $y=a(x^2-x_0^2)$ we also obtain

$$\frac{dy}{dx}=2ax\qquad\text{hence}\qquad ds=\sqrt{1+4a^2x^2}\,dx\ .$$

But the inclination $\frac{dy}{dx}$ being ordinarily small compared to 1, we can, without appreciable error, replace the quantity under the radical by $(1+2a^2x^2)^2$ which differs from it only by $\frac{1}{4.}\left(\frac{dy}{dx}\right)^4$. We thus find that the element ds is slightly less than $(1+2a^2x^2)\,dx$. By integrating, we conclude that s, from the origin up to the abscissa c, is slightly smaller than $c+\frac{2}{3}a^2c^3$.

In the first case, the value of a is $\frac{p}{c^2}$ and in the second $\frac{p}{c^2-\frac{1}{4}\delta^2}$; but $\frac{1}{4}\delta^2$ can be neglected compared to c^2; so that the approximate expression of the length of the half-chain is in both cases

$$s=\left(1+\frac{2}{3}\frac{p^2}{c^2}\right)c$$

60. If the length s undergoes a small increase due to the effects of temperature variations, for example, and we wish to calculate the corresponding increase in the sag p, we will achieve this by regarding these increases as infinitesimal and differentiating the above equation, which amounts to neglecting the square of the increase of p. We will obtain

$$ds=\frac{4}{3}\frac{p}{c}\,dp\qquad\text{hence}\qquad dp=\frac{3}{4}\frac{c}{p}.\,ds\ ;$$

since s differs little from c, if we put this equation in the form

$$\frac{dp}{p}=\frac{3}{4}\frac{c^2}{p^2}.\frac{ds}{s}\ ,$$

we see that the proportional increase of p is much larger than that of s.

§:2 *Polygonal systems of solid bodies with simple articulations*

61. Imagine that several solid bodies touch each other at points which cannot separate: no obstacle, no friction otherwise preventing the rotational movements compatible with this condition. We will first stop at the case where each body touches only two others, each by one point, and the system will then be polygonal.

One can conceive each articulation as produced by the assembly called a knuckle joint, consisting of a small sphere adhering to one of its bodies, and enveloped by a portion of a hollow sphere belonging to the other body. (The assembly called a hinge allows rotation only perpendicular to the axis, and consequently would not leave the system all the freedom of movement assumed here)

The aim is to study the conditions of equilibrium of such a system.

62. First, consider separately one of the solid bodies of the system. It is acted upon by any external forces $F_1, F_2, F_3, \ldots\ldots$ and in addition by the equally external pressures it receives from the two bodies with which it is linked; and these pressures N', N'' pass through the centers C, D of the articulation spheres, (fig. 5) since there is no friction.

The six conditions of equilibrium are fulfilled by the set of these forces, and are sufficient since it concerns a solid body.

One of these conditions can be made independent of the pressures N', N'': it consists in that the algebraic sum of the moments of the forces $F_1, F_2, F_3, \ldots\ldots$ with respect to the line CD is zero, which, if we take CD as the z-axis, will be expressed by the notation

$$\sum M\{F\}_z = 0$$

This condition amounts to saying that the forces $F_1, F_2, F_3, \ldots\ldots$ can be reduced to two *equivalents* applied at points C and D. Indeed we know (48) that in all cases we are free to make one of the equivalents pass through the first point; in the present case the moment of the second equivalent being zero, because of $\sum\{F\}_z = 0$, this force lies in the same plane with the axis CD, and if we join its point of application, whatever it may be, with the points C and D by two straight lines, we can always decompose this force into two passing through these points. We see from this that if the articulation centers C, D are maintained by an infinite resistance, the condition $\sum M\{F\}_z = 0$ will suffice to ensure equilibrium.

The other five equilibrium conditions, which must always be satisfied by all external forces, including the pressures N', N'', will provide relations between these pressures and the forces F or their equivalents.

To this end, if through C we draw two axes Cx, Cy mutually perpendicular and to Cz, and we denote by $N'_x, N'_y, N'_z, N''_x, N''_y, N''_z, F_x, F_y, F_z$, the projections of the forces onto the axes, and we set $CD = l$, we have five other equations involving the pressures.

$\sum M\left(F\right)_x + l\,N_y'' = 0$, $\sum M\left(F\right)_y + l\,N_x'' = 0$, $\sum F_x + N_x' + N_x'' = 0$, $\sum F_y + N_y' + N_y'' = 0$, $\sum F_z + N_z' + N_z'' = 0$; the first four suffice to determine N_x' , N_y' , N_x'' , N_y'' ; but the last gives only the sum $N_z' + N_z''$.

Take for example a body supported by the two centers of a lathe. Knowledge of the forces *F*, such as the weight of the body, the tension of the cord, the pressure of the tool, will determine the pressures perpendicular to the axis, but not the pressures in the direction of this axis. It is conceivable, in fact, that one could, without disturbing equilibrium, vary the latter, provided their algebraic sum remained the same.

63. Let us now examine the equilibrium of a polygonal system of solid bodies articulated among themselves.

Each body considered separately is in equilibrium, (remark analogous to the first of No. 45) therefore (62) the forces *F* acting on this body, apart from its articulations, with the two contiguous bodies, have the algebraic sum of their moments zero with respect to the line joining the centers of these articulations; and since this body is supposed rigorously solid, the actual forces *F* can be replaced without disturbing equilibrium by two equivalents applied to these two articulations.

By making the same modification of the external forces for each solid body of the system, one would transform it into an assembly of rigid bodies which would be subjected to no force except at their articulation points; from then on the shape and mass of each body would become indifferent to the question, provided the distance between these two articulations remained the same. One could therefore consider the system as composed of rigid straight weightless rods, forming a polygon whose vertices alone would be subjected to external forces. In a word, one would thus have a system which would differ from a proper funicular polygon only in that the sides would be rigid and could consequently be compressed longitudinally, and support what might be called a negative tension.

The transformation we have just spoken of is frequently employed by authors; it can sometimes be convenient, but it is not indispensable, and a few simple examples will show how one can, without recourse to it, proceed in the study of the equilibrium conditions of a system of articulated bodies, and in the calculation of the pressures occurring at the articulations.

64. Two bodies linked together by the movable articulation *M*, (fig 6) and able to rotate about the fixed articulations *A*, *B*, are acted upon by vertical forces *F*, *F′*, intersecting the lines *AM*, *MB*. The pressures at *A*, *B* and *M*, which it is important to know to ensure afterwards whether the pieces of the system can resist the forces they support, are required. For equilibrium, the plane *AMB* must be vertical; for the entire system is in equilibrium under the action of the forces *F*, *F′* and the two equally external pressures it receives at the two extreme articulations *A B*; whence it is easy to conclude that the forces *F*, *F′* intersect the line *AB*. Let *N, N′*, be these pressures at *A* and *B*, whose horizontal and vertical projections are

$N_x\ N_y$, $N'_x\ N'_y$, positive or negative quantities depending on whether the components they represent act in the direction of the axes A_x, A_y or in the opposite direction.

That established, we will make two remarks:

Firstly, the body MB is in equilibrium under the force F' and the pressures at M and B: therefore the moment with respect to a horizontal axis projected at M gives (see the figure for the meaning of letters)

$$[\ 1\ ] \qquad F'_B\, z + N'_x\, b' - N'_y\, a' = 0$$

in which N'_x and N'_y must be taken with their signs

Secondly, the set of the two bodies is in equilibrium under the forces F, F', N, N'.

Therefore the moments with respect to a horizontal axis through A give

$$[\ 2\ ] \qquad F\ x + F'\ x' + N'_x\, b - N'_y\, c = 0$$

and the condition of the nullity of the resultant of translation is moreover expressed by the two equations

$$[\ 3\ ] \text{ and } [\ 4\ ] \qquad N_x + N'_x = 0\ ,\ N_y + N'_y - (F + F') = 0$$

The above four equations suffice to determine the four quantities N_x, N_y, N'_x, N'_y and consequently the pressures N and N', their direction and their sense.

At point M each of the bodies exerts a pressure on the other. Let N'' be the pressure that body AM exerts on body MB.

We will have [5] and [6] $N''_x + N'_x = 0\ ,\ N''_y + N'_y - F' = 0$.

One could have, instead of equation [2], obtained another, by noting that, according to equation [1], the forces F' and N' have a resultant passing through M; therefore they can be transported parallel to themselves to this point without changing the equilibrium conditions.

Then the equation of moments with respect to the axis through A becomes

$$F_x + F'_a + N'_x\, b - N'_y\, a = 0$$

But this equation is implicitly contained in equations [1] and [2] which give it by subtraction.

One would also have arrived at a system of equations equivalent to [1], [2], [3], [4], by considering the equilibrium of body MB about M, then that of body MA about M, and finally the nullity of the resultant of translation, expressed by equations [3] and [4].

Remark. After having found, by means of the six equations of the first degree of No. 64, the six quantities $N_x, N_y, N'_x, N'_y, N''_x, N''_y$ as functions of the forces F, F', and of the various coordinates, if we were to change the signs of the forces F, F', which amounts to supposing that they act in the positive direction of the axis A_y, we would find for the projections of the pressures the same absolute values, but opposite signs. This would occur if the heavy system indicated by fig. 6 were replaced by another situated symmetrically below the horizontal line A_x.

One will treat the same way the case of fig. 7 where the articulation points A and B are on a vertical line.

Equilibrium of AM about M:

$$N_x\, b' + N_y\, b + P\,(b-a) - F\,(c-b) = 0$$

Equilibrium of the system about B:

$$N_x\, b + Pa + P'\, a' + Fc = 0$$

Translational equilibrium:

$$N_x + N'_x = 0 \;\; ; \;\; N_y + N'_y + F + P + P' = 0$$

calculation of the pressure N'' that BM receives at M from the upper body:

$$N''_x + N'_x = 0 \;\; ; \;\; N''_y + N'_y + P' = 0$$

68. If the articulations AB were on a vertical cylinder held by a collar at C (fig. 8) and by a footstep bearing at D, this collar, without friction, would exert a horizontal pressure X, and the footstep bearing an oblique pressure Q. We would have:

Moments with respect to D: $XK + Pa + Fc + P'\, a' = 0$

Translation: $Q_x + X = 0$; $Q_y + (F + P + P') = 0$;

three equations to determine X, Q_x, and Q_y .

69. <u>System of four bodies symmetrical, in dimensions and forces, with respect to the vertical</u> $M'C$. (fig. 9). The external forces, apart from the articulations A, B, are supposed vertical.

The pressure of MM' on $M'M''$, and that of $M'M''$ on MM', at articulation M' are necessarily symmetrical, but moreover they are equal and directly opposite; therefore they are horizontal.

It therefore suffices to study the equilibrium of the system of fig. 10 in which N' is an unknown horizontal force, P and P' the vertical resultants of the forces acting on bodies AM, MM'.

The equilibrium of MM' gives the moment equation

$$N'\, h' = P'\, \xi$$

The equilibrium of the set of the two bodies gives

1° Moments $\qquad N'\, h = P_x + P'_x$

2° Translation $\qquad Q_x = N' \;\; ; \qquad Q_y = P + P'$

From the two moment equations we obtain:

$$\frac{P_x + P'_x}{h} = \frac{P'\, \xi}{h'}$$

the condition equation which will serve to solve various problems relating to the equilibrium of the system in question.

This equation would remain the same if the forces P, P' changed sign, i.e., reversed sense, or if, the forces always acting from top to bottom, the figure were replaced by another symmetrical to the first with respect to A_x .

70. The last equation not being supposed satisfied, we ask what horizontal force X would need to be applied at M to one or other of the two bodies articulated at this point to obtain equilibrium.

By the same considerations we have

$$N' h' = P' \xi$$

$$N' h = P_x + P'_x + X b$$

hence

$$X = \left(\frac{P' \xi}{h'} - \frac{P_x + P'_x}{h} \right) \frac{h}{b}$$

X is the tension of a thread whose weight would be negligible, and which would join the points M, M'', (fig. 9).

71. One would proceed analogously for more complicated cases.

Let there be any number of consecutive solid bodies linked by articulations $M', M'', M''' \ldots M^n$ the two extreme bodies are moreover constrained to move only by rotating about the articulations A, B whose centers are fixed. The $n + 2$ points $A, M', M'' \ldots M^{(n)}, B$ lie in the same plane. The $n + 1$ bodies are acted upon by known external forces $F_1', F_2' \ldots F_1'', F_2'' \ldots F_1''', F_2'''$ situated in the same plane, but otherwise arbitrary. We ask to verify whether equilibrium exists, and in the affirmative case, to find the pressures on the supports A, B and the mutual pressures at M', M'', M'''.

Firstly, proceeding as in No. 64 we will verify the equilibrium of the system of the two contiguous bodies $AM', M'M''$; and in the affirmative case we will determine the pressure N exerted by support A, that N' which body AM' exerts on its neighbor, and that N'' which $M'M''$ exerts at M'' on $M''M'''$.

Secondly, the body $M''M'''$ being in equilibrium under the action of the forces N'', F_1''', F_2'''... and of the force it receives at M''' from the contiguous body, we will verify this condition either by the successive composition of the known forces whose resultant must pass through M''', or by the property of moments taken with respect to M'''; and if it is satisfied, the resultant of translation of the forces F_1''', F_2''', ... N'' will be precisely the pressure N''' that the considered body, $M''M'''$, exerts at M''' on its neighbor $M'''M^{IV}$.

Thirdly, the body $M'''M^{IV}$ having to be in equilibrium under the action of the known forces F_1^{IV}, F_2^{IV}, ... and N''', and of the force exerted at M^{IV} by the contiguous body $M^{IV}M^V$, we will proceed similarly to verify this equilibrium and determine the pressure N^{IV} exerted at M^{IV} on body $M^{IV}M^V$.

And so on up to the last part of the articulated system.

§: 3 Of completely constrained systems.

72. A system of solid bodies linked together is said to be a completely constrained system, when these bodies are so united among themselves and with other fixed bodies that the motion of any point of the system completely determines the motion of all the others, and that each of these points can only move on a certain straight or curved line.

Example: various balances, the so-called weighbridge, that of Quintenz, that of Roberval; two beams linked by a parallelogram; gears.

73. When, in the motion that such a system can take, it is permissible to neglect frictions, i.e., the work of the mutual actions of the bodies among themselves and that of the reactions of the fixed bodies, the most general and ordinarily the most convenient principle for finding the condition of equilibrium is the theorem of virtual work.

It consists in that for equilibrium to occur, it is necessary and sufficient that the external forces be such that their total work would be zero if the possible motion of the system were supposed to take place.

The demonstration of this very important theorem results from that which was given (45) with a view to establishing the six general conditions of the equilibrium of a material system.

1° The above-stated condition is necessary: indeed it holds for each solid body of the system by joining to the forces proper to it the mutual actions it receives from the other neighboring bodies. Now, by hypothesis, the total sum of the work of the mutual actions of the bodies among themselves is zero. Therefore if, after having set down for each body the equation expressing the condition in question, we sum all the equations, we will find that the total work of the external forces alone is also zero.

2° This condition is sufficient: because if it is satisfied it is impossible that the system supposed at rest should set itself in motion, since there would be acquired living power without work of forces, which cannot be (39).

The application of the theorem of virtual work to the systems indicated in No. 72 reduces to questions of geometry.

74. Weighbridge (fig. 11) A platform AA on which the body to be weighed is placed, is hinged at a point B invariably linked with a lever ED whose fixed axis is at D.

Another lever HF whose fixed axis is at I, is joined to the platform AB and the lower lever ED by two rods GK, FE articulated at their ends. At the end H a small pan is suspended. The weights of the various parts of the apparatus are adjusted so that the platform and pan being empty, the rods GK, FE are vertical, as are the diametral planes of the knife-edges of the various articulations. This position of empty equilibrium is also that of the equilibrium which occurs when the platform and the pans are loaded with weights Q and P. For this purpose the articulations are situated so that (as will be seen) in a virtual motion of the apparatus all points such as $K\,A\,B\,A$, invariably linked to the platform, rise or fall by equal amounts, so that while point B describes a small arc of a circle about D, the plane AA of the platform remains

horizontal. By means of this condition, the virtual work of the weight of the object placed on the platform is the same as if this weight were applied at *K*, and the apparatus functions like a steelyard *HIG* and we have $P : Q :: I_g : I_b$.

It only remains to see how the condition of translational motion of the platform is satisfied. Through *D* and *I* draw horizontals meeting the verticals *Bb, Ef, GK,* at *b, e, f, g*, and suppose these horizontals linked to the levers *DE*, *IF*. We easily see: 1° that the virtual displacement of *B*, projected vertically, is equal to the elementary vertical arc described by point b which we will call ζ; 2° that the vertical displacements of *e* and *f* are equal to each other, being respectively equal to those of *E* and *F* projected vertically; thus the virtual displacement of *f* is $\zeta \frac{\overline{De}}{\overline{Db}}$; 3° that the virtual displacement of *K* projected vertically is equal to those of *G* and *g*, and that the latter, compared to that of *f*, is $\zeta . \frac{\overline{De}}{\overline{Db}} . \frac{\overline{Ig}}{\overline{If}}$. By equating this last quantity to ζ we find the required relation

$$\frac{\overline{Db}}{\overline{De}} = \frac{\overline{Ig}}{\overline{If}}$$

75. Drawbridge. (fig. 12) *A* is the axis of rotation of the bridge deck, *B* the attachment point of the chain *BC*, *G* the center of gravity of the deck and *P* its weight. *D* is the axis of rotation of the upper movable frame, *G′* the center of gravity of all this frame, *Q* its weight, *C* the attachment point of the chain *CB* at the end of the jib. Care is taken that quadrilateral *ABCD* is a parallelogram. Then *AB*, *CD* remain constantly parallel during their motion. Therefore the work of the weight *P* applied at *G* is the same as would be that of an equal weight *P* applied at point G_1 situated with respect to *DC* as *G* is with respect to *AB*. That established, for equilibrium to subsist in any position, abstraction made of the weight of the chain *BC* and of frictions, it is necessary and sufficient that the line G_1G' pass through *D*, or in other words, that *GA* be parallel to *DG′* and that we have

$$Q \ : \ P \ :: \ GA \ : \ DG'$$

§: 4 Of systems with alternating articulations.

76. Let *ABCD* (fig.13) be a solid body, resting on another body *LM* by a face whose projections are *AB* and *a e b c d*, without being able to detach itself by sliding, but with the possibility of rotating about any point or edge of the perimeter of this face. The body *AC* being acted upon by various forces outside the plane *AB*; for rest to subsist, these forces must be in equilibrium with the resistances that body *LM* exerts on body *AC*, and which all, starting from points on the face *AB*, are directed towards body *AC*. Therefore if, for any point (*A, a*) of the perimeter of the contact face, we imagine an axis (projected at *A*), which does not cross this face, it will be necessary that the sum of the moments with respect to this axis, of forces such as $F_1, F_2 \ldots$ be of the same sign (except for some which are zero, which occurs when the forces *F* have a resultant passing through *A*) so that the body *AC* would tend, under the action of these forces, to rotate in the opposite direction to that determined by the moment about that

axis. The fulfillment of this condition for all points of the perimeter of the contact face is necessary and sufficient for equilibrium to exist.

77. Particular case: When the forces acting on body *ABCD*, independently of the resistance of *LM*, have a resultant, the preceding condition amounts to saying that this resultant must be directed from the body towards the contact face, and meet its plane inside or at least on the perimeter of the smallest convex polygon surrounding the points of contact.

78. Equilibrium in the case of inequality of moments, indicated in No. 76, has this remarkable character that the forces $F_1, F_2, F_3, \ldots\ldots$ can, within certain limits, vary independently of one another without the body *AC* to which they are applied taking any appreciable motion; this is because the reactions of the resisting body *LM* modify themselves so as to satisfy the conditions of equilibrium; whereas in systems with simple articulation, the slightest change in the intensity or direction of one of the external forces generally produces motion.

Equilibrium in which forces other than the reactions of supports can thus vary plus or minus up to a certain point without producing motion, constitutes stability properly so-called.

79. When the forces F have a resultant (77) the reactions of body *LM* also have one, which is equal and opposite to the first; but it does not follow that one can determine the intensity and direction of these reactions for each given portion of the contact surface.

In rational mechanics which admits the existence of perfectly hard and sometimes polished bodies, it is concluded from this hypothesis that for the equilibrium of body *AC* it is necessary that the resultant R of the forces F be perpendicular to the plane *AB* of the contact points; and if these points are reduced to three, one easily determines the normal reaction of each of these points, but if there are more than three points of contact, the problem posed under the hypothesis of the invariability of shape of the body is indeterminate, whereas it is determined in the actual state of bodies by virtue of their compressibility and their elasticity.

80. One can remove the indeterminacy in question with the help of an hypothesis whose exactness is sufficient for practice. To state it and explain its consequences, consider a simple case. Let AA' (fig. 14) be the polygonal face by which the movable body rests on another; and suppose that the first is acted upon by various forces F whose resultant R is perpendicular to $A'A''$. Conceive that, before the application of these forces and when the body was in its natural state, a plane $B'B''$ parallel to $A'A''$ and very close was drawn in the solid body. This being so, it is supposed that the molecules which were in the plane $B'B''$ approach those occupying the plane $A'A''$, but in such a way that both are respectively found in two planes b'b", $A'A''$; finally it is supposed that the reactions which occur at the various points A', A, A'', being referred to unit area, are proportional to the amounts $B'b', B''b''$ by which the distances $A'B'$, AB, $A''B''$, initially equal, are decreased. Thus setting $B'b' = h'$, $B''b'' = h''$ and in general $Bb = h$, if we denote by ω a superficial element at *A*, and by p the pressure per unit area at this point, we will have for the total pressure on this element

$$\omega\, p = K\ h\ \omega$$

K being a constant. Now $h\,\omega$ is the volume of the cylinder having base ω and height Bb. Therefore we can take all such volumes as representing the pressures on the various superficial elements of $A'A''$; from which it follows that the resultant of all these pressures is represented by the total volume of the truncated prism $B'b'b''B''$ and passes through its center of gravity. And as this resultant is directly opposite to R, it follows that R passes through this same center of gravity. Thus, to know the distribution of the various reactions from A' to A'', given R, the question reduces to finding this plane $b'b''$ such that the truncated prism has its center of gravity on the extension of R.

81. If, for example, the contact face is a parallelogram of which $A'A''$ is one side, and the force R lies in a plane which bisects the sides projected at $A'A''$, the center of gravity of the truncated prism $B'b'b''B''$ will be projected at G, the center of gravity of the trapezoid $B'b'b''B''$. That established, let $B'G = a$, $B'B'' = l$, we will have, by the known theory of moments,

$$(h' + h'')\,a = \frac{1}{3}\,l\,(h' + 2\,h'') \quad \text{hence} \quad \frac{h'}{h''} = \frac{2 - 3\,\frac{a}{l}}{3\,\frac{a}{l} - 1} \qquad [\,1\,]$$

According to the previous number, denoting by ω the total area of the face $A'A''$, the total pressure P will be represented by $\Omega\left(\frac{h'+h''}{2}\right)$, while the pressures, per unit area, along the edges A', A'', pressures which we will designate by p', p'' will be represented by h' and h''. So we have

$$p' : P :: h' : \Omega\left(\frac{h' + h''}{2}\right) \quad \text{or} \quad p' = \frac{2}{1 + \frac{h'}{h''}} \cdot \frac{P}{\Omega} \qquad [\,2\,]$$

similarly

$$p'' = \frac{2}{1 + \frac{h'}{h''}} \cdot \frac{P}{\Omega} \qquad [\,3\,]$$

Equations [1], [2], [3] giving the extreme pressures p', p'', one can easily deduce, if needed, the pressure at any point A.

82. The nature of the question requires that the quantities h' and h'' be both positive unless one of them is zero, which requires according to [1] that one have simultaneously

$$2 \geq 3\,\frac{a}{l} \quad \text{and} \quad 3\,\frac{a}{l} \geq 1 \quad \text{or} \quad \frac{a}{l} \leq \frac{2}{3} \quad \text{and} \quad \frac{a}{l} \geq \frac{1}{3}$$

In the particular case where one would have $a = \frac{1}{3}\,l$ we would find

$$h'' = 0\,, \quad p'' = 0 \quad \text{and} \quad p = 2\,\frac{P}{\Omega}\,.$$

Thus on the edge A'' the farthest from R, the pressure would be zero, while along the edge A', the closest, the pressure per unit area would be double the average pressure per unit area over the extent $A'A''$.

Now if a $< \frac{1}{3} l$, it is natural to admit that the total pressure is distributed over a length $3a$ instead of l. Let b be the dimension of the contact face, perpendicular to $A'A''$; we must then in the proportion [2] put for ω the product $3a\ b$ at the same time as $h'' = 0$, from which results the maximum pressure per unit area

$$p' = \frac{2}{3}\frac{P}{ab}.$$

83. The following question may give rise to an application of this theory.

Two solid pieces AM', $M'B$, rest on two invariable planes AA', BB', and buttress each other mutually along the face MM'. Knowing the resultants F, F' of the forces acting on these pieces, independently of the mutual pressures at MM', and of the reactions of the supports AA', BB', we ask what these pressures are in the simple case where the forces F, F' are in a plane perpendicular to the contact faces AA', BB', MM', and passing through the centers of figure of these faces.

If we admit that the resultants N, N'' of the reactions of the supports AA', BB', and that N' of the mutual pressures on MM', pass through points C, D, E taken arbitrarily on the faces, the question becomes analogous to that of No. 64 and is solved without difficulty. One can moreover determine the pressures per unit area, near the edges A, A', M, M', B, B'. But if we impose the condition that these pressures must not exceed a certain limit, the choice of points C, D, E, though still indeterminate, becomes much more restricted, because these points must not be too close to the edges, unless at a distance determined by a trial and error procedure.

84. The same considerations are found in all questions relating to the stability of systems with alternating articulations. The following example is susceptible to numerous applications.

Let there be a system $E\ I\ i\ I_1\ E_1\ e\ E$ (fig. 15) of pieces symmetrical, in dimensions and forces, with respect to the vertical line $e\ i\ V$.

These bodies are assembled among themselves and with the lower fixed bodies on which the system rests, in such a way that the extreme bodies can alternately rotate about the edges I, E, I_1, E_1; that the bodies $e\ A'$, $e\ A'1$ can partially detach themselves by rotating about either of the edges i e; that the bodies $e\ A'$, $B'A''$ are likewise linked by an alternating articulation at A' and B'; and so on. We confine ourselves to the case where the resultants F', F'', . . . of the known external forces lie in a vertical plane, before and behind which the figure of the system is symmetrical, and we propose 1° to find the conditions of stability of the system, 2° to determine the pressures of the pieces among themselves and on the two supports.

First, it will be seen as in No. 69 that the resultants of the reciprocal pressures which occur at ie are horizontal, and that one can consider that the bodies comprised between IE and ie, the latter plane being acted upon at an unknown point c by an unknown force N.

Thus the body $A'e$ is in equilibrium under the action of forces N and F', and of the reactions it receives from body $A'B''$, which have their resultants passing through a point C' between A' and B.

Generally, the assembly iA_nB_ne of any number of parts of the system, comprised between the points ie, A_nB_n,, is in equilibrium under the action of the external forces which are 1° the horizontal force N passing through c between i and e; 2° the known forces F', F''....., $F^{(n)}$; 3° the resultant (passing through $C^{(n)}$ between $A^{(n)}$ and $B^{(n)}$) of the reactions or pressures that the body below the plane $A^{(n)}B^{(n)}$ exerts on the body above this plane.

It follows that if we consider the moments of these forces with respect to the line projected at $A^{(n)}$, the moment of the force N is equal to the sum of the moments of the forces F', F'', ..., $F^{(n)}$ with respect to the same axis. This is what we will write thus

$$N\,\zeta^{(n)} > \sum M\left\{F',\,F''\ldots.\,F^{(n)}\right\}_{A^{(n)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots [\,1\,]$$

calling $\zeta^{(n)}$ the vertical ordinate of point $A^{(n)}$ below the horizontal plane passing through c.

Similarly, the whole system comprised between EI and ei is in equilibrium under the action of force N, all forces F, and a resultant passing through C of the pressures exerted from bottom to top by the support, on the body above the plane EI; from which we will easily conclude that if we take moments with respect to the outer edge E, we will have

$$\sum M\left\{F\right\}_{E} > N_{\eta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [\,2\,]$$

calling η the vertical ordinate of E below the horizontal plane passing through c.

The two inequalities [1] and [2] can be regarded as expressing that if the system consisted of 2 solid pieces $E\,I\,A^{(n)}\,B^{(n)}$, $B^{(n)}\,A^{(n)}\,i\,e$, the movement which would consist in the subsidence of the joint ie and the rotation of the lower piece about E, could not take place.

Multiplying the two similar inequalities [1] and [2] we obtain:

$$\zeta^{(n)} \sum M\left\{F\right\}_{E} > \eta \sum M\left\{F',\,F''\ldots.\,F^{(n)}\right\}_{A^{(n)}} \quad \ldots\ldots\ldots [\,3\,]$$

Now to eliminate from this inequality the heights $\zeta^{(n)}$, η which always remain indeterminate, we note that if we call $z^{(n)}$ and h the ordinates of points $A^{(n)}$ and E below the horizontal plane passing through e, we have

$$\frac{z^{(n)}}{h} > \frac{\zeta^{(n)}}{\eta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [\,4\,]$$

since the two terms of the first fraction exceed those of the second by the same quantity $c\,e$. Multiplying the two similar inequalities [3] and [4] we find

$$\frac{\sum M\left\{F\right\}_{E}}{h} > \frac{\sum M\left\{F',\,F'',\ldots\,F^{(n)}\right\}_{A^{(n)}}}{z^{(n)}} \quad \ldots\ldots\ldots\ldots [\,5\,]$$

The first member of this inequality has a single value; the second member on the contrary takes various values according as it is applied to one of the points A', A'' . . . of the intrados of the system.

Inequality [5] can be summarized by saying that the force expressed by the quotient

$$\frac{\sum F \{F\}_E}{h}$$

must be greater than the greatest of the values taken by the quotient

$$\frac{\sum F \{F', F'' \dots F^{(n)}\}_{A^{(n)}}}{z^{(n)}}$$

when the index n varies.

We have just considered the movements $A^{(n)}$ and E, one on the intrados, the other on the extrados; and the two inequalities [1] and [2] allowed us to eliminate N. We will obtain an analogous result by considering the axes of moments $B^{(n)}$ and I, the first on the extrados, the second on the intrados. We see by reasoning similar to the preceding that we have

$$N\ \psi^{(n)} < \sum M \left\{F', F'' \dots F^{(n)}\right\}_{B^{(n)}} \quad \dots\dots\dots [\ 1^{\text{bis}}\ ]$$

calling $\psi^{\{(n)\}}$ the ordinate of $B^{(n)}$ below the horizontal passing through c;

$$\sum M \{F\}_I < N\ \chi \dots\dots\dots\dots\dots [\ 2^{\text{bis}}\ ]$$

calling χ the ordinate of I below the same horizontal;

$$\frac{y^{(n)}}{k} < \frac{\psi^{(n)}}{\chi} \quad \dots\dots\dots\dots\dots [\ 4^{\text{bis}}\ ]$$

calling $y^{(n)}$ and k the ordinates of $B^{(n)}$ and I below the horizontal passing through i.

Multiplying these three inequalities together, we find

$$\frac{\sum M \{F\}_I}{k} < \frac{\sum M \left\{F', F'' \dots F^{(n)}\right\}_{B^{(n)}}}{y^n} \quad \dots\dots [\ 5^{\text{bis}}\ ]$$

i.e., that the force expressed by the first member of this inequality must be smaller than the smallest of the values taken by the second member when the index n varies.

Inequalities [5] and [5^{bis}] are the conditions of stability sought, when supposing the bodies of the system sufficiently resistant, so that the forces to which they are subjected, if one of these inequalities became an equation, that would mean that the system would be in unstable equilibrium, and that the point $i\ e$ being on the point of opening, the pressure N would act either at e (if [5] became an equation), or at i (if it were [5^{bis}]). This is what would be found directly by taking up the previous reasoning in this hypothesis.

85. The search for the resultant pressures supported by the points is a problem which, in the case of stability of the system, and in the hypothesis of unlimited resistance of materials, has an infinite number of solutions. It would become determined if we knew two points of the polygon $cC'C''C^{(n)}C$ of the centers of pressure, for example c and $C^{(n)}$. Indeed we would deduce from it the value of N by the equation of moments (see the figure)

$$N\ .\ \overline{cD}^{(n)} = \sum M \left\{F', F'' \dots F^{(n)}\right\}_{C^{(n)}} \dots\dots [\ 6\ ]$$

If the two known points were $C^{(n)}$ and C, we would have, besides the preceding equation

$$N \,.\, \overline{cD} \;=\; \sum M \{F\}_C$$

hence by subtraction,

$$N \,.\, \overline{DD^{(n)}} \;=\; \sum M \{F\}_C \;-\; \sum \left\{F\,',\, F\,'',\, \ldots\, F^{(n)}\right\}_{C^{(n)}}.$$

The force N being thus determined, point C' is found graphically or by calculation,

on the resultant of the known forces N and F', point C'' on the resultant of the known forces N, F', F'' and so on up to point C.

86. It is on this observation that the following procedure is founded, the idea of which belongs to Mr. Mery (*Annales des Ponts and Chaussées*, Jan. 1840), for verifying the stability of a barrel vault, having regard to the limited resistance of materials. One chooses point $A^{(n)}$ in such a way that it is approximately the one which gives the greatest value to the quotient

$$\frac{\sum M \left\{F\,',\, F\,''\, \ldots\, F^{(n)}\right\}_{A^{(n)}}}{x^{(n)}}$$

forming the second member of inequality [5]. This quotient, which is calculated approximately, gives a value less than N, (which is seen by multiplying inequality [1] by the similar inequality $Z^{(n)} > \zeta^{(n)}$), but which generally differs little from it. Substituting it in the final formula of No. 82 in place of P, and also putting for p' the maximum pressure per square meter, to which the considered materials can be subjected, we find the value

$$a \;=\; \frac{2}{3}\,\frac{N}{bp'}$$

of the smallest magnitude that can be given to the distance $e\,c$.

87. On the other hand, by transporting parallel to themselves to the same point of application the forces N, F', F''. . . . . $F^{(n)}$ and calculating their resultant, we will likewise have an approximation of the total pressure on the joint $A^{(n)}\,B^{(n)}$; from which we will deduce the smallest magnitude that the distance $A'^{n)}\,C^{(n)}$ can have.

88. Then taking points C and $C^{(n)}$ so as to satisfy these two conditions: we will calculate more precisely the force N by formula [6] of No. 85, and if it differs noticeably from the first calculated one, we will modify points C and $C^{(n)}$ accordingly. This done, we will determine [85] other points of the polygon of pressure centers, notably point C, which, having regard to the total pressure on joint $I\,E$, which we will have calculated in the same way as that of joint $A\,B^{(n)}$, (87), must be distant from edges F_1, I, at least by a distance which can also be calculated.

If this condition is satisfied, it will be necessary to ensure that it is also satisfied for all the other joints.

Chap. 3. On the resistance of materials used in constructions.

§:1 *Prisms stretched or compressed parallel to their length.*

89. A prism acted upon at its ends by forces parallel to its length and uniformly distributed over each base lengthens or shortens by an amount which, when rest is reestablished, follows a very simple law, as long as the intensities of the forces are not too considerable. If we call

L the original length of the prism

x its elongation

i the ratio $\frac{x}{L}$, i.e., the elongation or shortening per unit length

F the total force exerted on one of the bases of the prism

ω the area of this base or the area of the right cross-section of the prism

R the ratio $\frac{F}{\Omega}$ i.e., the force (tension or compression) exerted per unit area of the right cross-section of the prism;

the two quantities R and i are directly proportional and we consequently have

$$R = Ei \quad \text{or} \quad F = E\frac{\Omega}{L}x \quad \ldots\ldots\ldots\ldots\ldots\ldots [1]$$

E being a constant as long as the nature of the body does not change and as long as the force R does not exceed a certain value called the limit of elasticity.

For example, in the case of wrought iron, taking the meter and the kilogram as units, we have approximately

$$E = 2 \times 10^{10}$$

from which we will conclude that if the tension is 10 kg per square millimeter, or $R = 10 \times 10^{6}$, the elongation per meter i is $\frac{1}{2\,000}$ or $\frac{1}{2}$ millimeter.

For this iron the elastic limit corresponds to a tension varying from 12 to 15 kg per square millimeter, depending on the quality of the iron; thus the greatest value of R compatible with the above law is from 12×10^6 to 15×10^6, and the corresponding value of i is from 0.0006 to 0.00075.

90. These experimental results can be verified by suspending a body of known weight from the lower end of a vertical prismatic rod, fixed at the other end. To show how important it is not to let this body acquire an appreciable velocity when measuring the elastic limit, we will treat the following problem which we borrow from the *Introduction to Industrial Mechanics* by Mr. Poncelet, pages 385 and following.

A vertical prismatic rod is fixed at its upper end. A body M, whose weight is Q, being traversed by this rod, slides without friction parallel to its length, and has thus acquired a

velocity V_0 at the instant when, stopped by a catch at the lower end of the rod, it can descend no further without receiving from this rod a force opposed to that of gravity. Having said that, we ask from this instant the law of the movement of the body M considered as condensed at its center of gravity, neglecting the mass of the rod and admitting consequently that as it lengthens its tension increases uniformly throughout its length and follows the same law as if under this tension, it were at rest.

Let us call, at any instant, keeping the previous notations,

x the length traveled by body M and by the lower end of the rod, since the instant of their linkage,

v the velocity acquired by body M.

The variable force F that the rod exerts on this body has the expression

$$E \frac{\Omega}{L} x$$

according to the previous number, and the hypothesis just stated. Thus the work of forces Q and F applied to body M during the traversal of x is

$$\int_0^x (Q - F)\, dx \qquad \text{or} \qquad \int_0^x \left(Q dx - \frac{E\,\Omega}{L}\, x dx \right).$$

The living power acquired by body M by virtue of this work is

$$\frac{1}{2} \frac{Q}{g} \left(v^2 - v_0^2 \right)$$

we therefore have (89)

$$\frac{1}{2} \frac{Q}{g} \left(v^2 - v_0^2 \right) = Q\, x - \frac{E\,\Omega}{2\,L} x^2 \ldots\ldots\ldots\ldots\ldots\ldots [\,2\,]$$

To simplify, let l denote the elongation of the rod at the instant when its tension is Q; this is the elongation it would take and maintain if body M were suspended from it at rest. To determine it, we must set $F = Q$, and $x = l$ in equation [1], whence $\frac{E\,\Omega}{L} = \frac{Q}{l}$

Substituting in equation [2] we have

$$v^2 - v_0^2 = \frac{g}{l} (2\,l - x)\, x \ldots\ldots\ldots\ldots\ldots\ldots [\,3\,]$$

The maximum velocity corresponds to $x = l$ and $F = Q$, which could have been seen <u>*a priori*</u>, the maximum is given by the formula

$$V^2 = V_0^2 + gl$$

As long as V is positive, the rod lengthens: the maximum elongation X corresponds to $v = 0$ and is given by

$$X^2 - 2\,lX = \frac{l\,v_0^2}{g} \qquad \text{or} \qquad X = l + \sqrt{l^2 + \frac{l\,v_0^2}{g}}$$

the corresponding tension according to [1], is to Q in the ratio of X to l; it is therefore

$$\frac{Q\,X}{l} \qquad \text{or} \qquad Q\left(1 + \sqrt{1 + \frac{v_0^2}{gl}}\right)$$

Thus, for example, in the particular case where v_0 is zero, i.e., when the weight Q first attached to the rod without tension is suddenly abandoned without vertical velocity to the action of gravity, the rod takes, at the instant the body stops, a double elongation and a double tension of the values which would correspond to the state of rest. When v_0 is not zero, the maximum tension depends on the ratio of the height $\frac{v_0^2}{2g}$ to $\frac{1}{2}l$, and this latter quantity is, all other things being equal, proportional to L.

91. From the instant of maximum elongation, body M rises again, retracing the same values of x, the same velocities but in opposite direction. It therefore possesses an upward velocity equal to v_0 at the instant when the rod has returned to its natural state; in this position, if the body can slide along the rod without friction, it will rise to the height $\frac{v_0^2}{2g}$ due to v_0, then it will fall back. If on the contrary it is fixed at the end of the rod and the latter cannot flex, it will contract, exerting on body M a pressure from top to bottom whose intensity will be given by the formula

$$F = \frac{E\,\Omega\,x'}{L},$$

in which x' is the decrease in length of the rod. The action of the latter is therefore then, in all possible positions, given, for intensity and direction, by the formula

$$F = \frac{E\,\Omega\,x}{L}$$

previously introduced into the calculation, by changing the sign of x when body M is above the initial position. In this case formula [3] applies to all values of x, positive or negative; setting $v = 0$ in it gives the extreme values of x, namely:

$$l \pm \sqrt{l^2 + \frac{l\,v_0^2}{g}},$$

from which it follows that the amplitude of oscillations is

$$x\sqrt{l^2 + \frac{l\,v_0^2}{g}}.$$

Let A (fig. 16) be the origin of x;

$$AO = L \;;\; OB = OB' = \sqrt{l^2 + \frac{l\,v_0^2}{g}} = p$$

AM being a value of x, let us represent by z the variable OM, and transform equation [3] by replacing x by its value in terms of z. We will find

$$x = l + z \;\; ; \quad 2\,l - x = l - z$$

hence $\qquad v^2 - v_0^2 = \frac{g}{l}\left(l^2 - z^2\right) \qquad$ or $\qquad v^2 = \frac{g}{l}\left(p^2 - z^2\right)$

the velocity v is therefore represented by the ordinate MN or y of the circle constructed on BB' as diameter, multiplied by the coefficient $\sqrt{\frac{g}{l}}$.

If we seek the duration of the oscillations, we must set $v = \frac{dz}{dt}$ in the last equation, and derive from it

$$dt = \sqrt{\frac{l}{g}}\,\frac{dz}{\sqrt{p^m - z^2}}$$

hence by integrating

$$t = \sqrt{\frac{l}{g}}\; arc\left(\sin = \frac{z}{p}\right) + c$$

and taking this integral from $z = -p$ to $z = p$ we find the duration of a simple oscillation, going or returning

$$T = \pi\sqrt{\frac{l}{g}}\,,$$

quantity independent of the velocity v_0.

Mr. Poncelet performs this integration by an elementary geometric process. Let MM' be an infinitesimal space dz traversed during time dt, denoting the ordinate MN by y, we have

$$dt = \sqrt{\frac{l}{g}}\,.\,\frac{dz}{y}$$

Drawing $M'N'$ parallel to MN, and $N'P$ parallel to MM', and denoting by ds the infinitesimal arc NN', we have, by the similarity of triangles OMN, $NN'P$,

$$\frac{dz}{y} = \frac{ds}{p} \qquad \text{and consequently} \qquad dt = \sqrt{\frac{l}{g}}\,.\,\frac{ds}{p}$$

therefore the time taken to traverse MM' is expressed by ds or NN' multiplied by $\sqrt{\frac{l}{g}}\,\frac{.1}{p}$; therefore the traversal time of BB' is

$$\pi\, p \times \sqrt{\frac{l}{g}}\,.\,\frac{1}{p} \qquad \text{or} \qquad \pi\sqrt{\frac{l}{g}}$$

Example. $\qquad L = 10\ \text{m}\ , \quad \frac{Q}{\Omega} = (4 \times 10^6)\ \text{kg.}, \quad E = (2 \times 10^{10})\ \text{kg.}, \quad v_0 = 0$

we find $\qquad l = \frac{QL}{E\,\Omega} = 0^m,002\ ;$

maximum velocity $\qquad V = \sqrt{gl} = 0^m,142\ ;$

duration of a simple oscillation $\qquad T = 3{,}1416\,\sqrt{0{,}00204} = 0^s,045\ ;$

number of double strokes per second $\frac{1}{2T} = 11{,}1$.

92. The preceding considerations make it understandable that a load insufficient to exceed the elastic limit of a piece can produce this effect if an accidental cause imparts a velocity to it, and explain what is improperly called the influence or action of time. Thus a rod which loses its elasticity only under a load of 12 to 15 kg per square millimeter ($R = 12 \times 10^6$ to 15×10^6) is considered only as being able to carry a permanent load of 6 to 8 kg per square millimeter ($R = 6$ to 8×10^6) .

§:2 *Generalities on the flexure of prisms*

93. When a prism is acted upon by forces inclined with respect to its length, it bends, and it is admitted that the material elements which were originally in a section perpendicular to the edges still remain, after their displacement, in a plane normal to the curves which replace these straight edges.

Let *AB* (fig. 16) be a prism bent by external forces, and at rest. Let *aa′, bb′* be two infinitely close cross-sections, which in the natural state would be parallel and which in the bent state make an infinitely small angle. We suppose that the part *bb′B* of the body is acted upon by external forces *F* whose resultant is sensibly parallel to the plane *aa′,* situated in a plane which will be that of the figure and which, dividing the prism into two symmetrical parts, is the plane of flexure.

This part *bb′B* of the prism is acted upon, in addition to the forces *F*, by the molecular actions that the elements of section *aa′* exert on those of section *bb′*. We will generally denote these latter forces by *A*.

This set of two kinds of forces satisfies the six general conditions of equilibrium. Three of these conditions lead to important consequences.

94. 1° — By taking an *x-axis* perpendicular to *aa′,* we have

$$\sum F_x + \sum A_x = 0 \qquad \text{or, according to the hypothesis, ...} \quad \sum A_x = 0$$

therefore the forces A_x are some positive, or repulsive, others negative or attractive, therefore there is shortening in the elementary fiber *a′b′*, lengthening in ab, and an intermediate fiber *oo′* that is invariable.

95. 2° — The axis of moments can be chosen arbitrarily. Let the one projected at o be chosen. We have $\sum M\{F\}_o = \sum M\{A\}_o$

taking as positive the moments of attractive forces above *o*, and that of repulsive forces below the same axis.

This quantity $\sum M\{A\}_o$ is what is called the moment of elasticity of the prism, in section *aa′*, for the state of flexure resulting from the forces *F*.

It results from the moment equation that when the distances of the forces F from the axis o are very large compared to the transverse dimensions of the prism (the only case studied here), the absolute sum of the forces A_x is very large compared to the intensities of the forces F.

96. 3° — By taking a *y-axis* parallel to *aa′*, we have

$$\sum A_y + \sum F_y = 0$$

This sum $[\sum F_y]$ being the resultant of the forces F, it follows that the sum of the forces A_y, component parallel to the plane *aa′*, of the molecular actions of *aa′* on *bb′*, is very small compared to the absolute sum of the forces A_x.

It is for this reason that the plane *bb′* can be regarded as having deviated from its original position, relative to the plane *aa′*, by turning about o' without sliding; so that this point *o′* remains situated as it was relative to the plane *aa′*. It also follows that the forces A can be considered as normal to *aa′*.

97. Let $d\,\omega$ be a superficial element of *aa′*, v its distance from the axis *o*, positive or negative depending on whether $d\,\omega$ is below or above *o*. The normal force A that it exerts on section *bb′* being proportional to the shortening or lengthening of the fiber which has this element as its base, and this variation in length being itself proportional to v, we have

$$A = \varphi\, v\, d\,\omega$$

calling $\varphi\, d\,\omega$ the value of the force which would occur at unit distance where $\sum A$ extended over the whole section *aa′* is zero (94). Therefore

$$\sum \varphi\, v\, d\,\omega = 0 \qquad \text{or} \qquad \sum v\, d\,\omega = 0$$

hence the axis *o* passes through the center of gravity of its section projected at *aa′*.

98. The moment of elasticity $\sum M\{A\}_o$, which varies according to the nature of the piece and according to its state of flexure at the location of the section *aa′* considered, can be expressed in various ways.

1st expression For the element $d\,\omega$, the force A being $\varphi\, v\, d\,\omega$, its moment is $\varphi\, v^2\, d\,\omega$ whether this element is above or below the axis *o*. Therefore for the whole section we have

$$\sum M\{A\}_o = \varphi \sum v^2\, d\,\omega\,.$$

$\sum v^2\, d\,\omega$ is what is improperly called the <u>moment of inertia of the surface</u> *aa′* with respect to the axis projected at *o*. It is known that this axis passing through the center of gravity, the sum $\sum v^2\, d\,\omega$ is smaller than if it were taken with respect to any other axis parallel to the first(*). We will denote this sum for brevity by I; thus

$$\sum v^2\, d\,\omega = I$$

(*) This general property of moments of inertia is very easy to establish, especially in the case of plane surfaces. Let dω be a superficial element, v its distance from an axis passing through the center of gravity, y its distance from another parallel axis, k the distance between these two axes.
We have $y^2 = (v \pm k)^2$ and $\sum y^2\, d\,\omega = \sum v^2\, d\,\omega + k^2\, d\,\omega$ because of the equation $\sum v\, d\,\omega = 0$

99. *2nd expression* Let R be the quotient $\frac{A}{d\,\omega}$, i.e., the resistance (tension or compression) per unit area that the element $d\,\omega$ placed in section aa' at distance v from the axis projected at o exerts on the corresponding element of section bb' to oppose either its approach or its separation. We therefore have

$$R = \varphi\, v \quad \text{or} \quad \varphi = \frac{R}{v},$$

hence moment of elasticity $\sum M\{A\}_o = \frac{R}{v} \sum v^2\, d\,\omega = \frac{R}{v} I$,

observing that, under the sign Σ, v has a general meaning, while in $\frac{R}{v}$, it is a specific distance, and the ratio $\frac{R}{v}$ is constant for a given state of flexure at the location of section aa'. R and v being thus proportional, their greatest values correspond so that if R' is the greatest resistance per unit area in section aa' and v' the greatest ordinate of the points of this section with respect to the axis projected at o, we have

$$\sum M\{A\}_o = \frac{R'}{v'} \sum v^2\, d\,\omega$$

or for brevity $$M = \frac{R'}{v'} I.$$

Finally, if we denote by R_1 the maximum resistance per unit area that the piece can exert without exceeding the elastic limit, we will have, relative to the various states of flexure of the piece, near section aa',

$$\text{limit of the moment of elasticity} = \frac{R_1}{v'} \sum v^2\, d\,\omega,$$

or for brevity $$M = \frac{R_1}{v'} I$$

It will be necessary, nevertheless, to distinguish whether at the point whose ordinate is v' the mutual action of sections aa', bb', is a pressure or a tension, because the value of R_1 is not always the same for the two cases. For example, for cast iron, when it comes to permanent forces, the values of R_1 that it is appropriate to adopt are 3×10^6 (or 3 kg per square millimeter) for tension, and 10×10^6 (10 kg/mm^2) for compression.

100. *3rd expression* of the moment of elasticity. Let c be the meeting point of aa' and bb', or the center of curvature of the piece at aa'. The element nn' whose section is $d\,\omega$ and whose distance from oo' is v, is shortened by $oo' - nn'$ and the repulsive force it exerts at n' is (89)

$$A = E\,\frac{oo' - nn'}{oo'}\, d\,\omega$$

now, $oo' : nn' :: oc : oc - v$ hence $\frac{oo' - nn'}{oo'} = \frac{v}{oc} = \frac{v}{\rho}$;

thus $$A = E\,\frac{v\,d\,\omega}{\varsigma}$$

therefore the total moment of the repulsive forces below the axis *o* with respect to this same axis is $E\sum v^2\,d\,\omega$, this latter sum being taken for the extent of the section below *o*.

Similarly for the element *mm′*, the attractive force it exerts at *m′* is

$$E\,\frac{mm' - oo'}{oo'}\,d\,\omega$$

now $mm' : oo' :: oc+v : oc$ hence $$\frac{mm' - oo'}{oo'} = \frac{v}{\rho}$$

giving *v* its absolute value. Therefore the attractive force is $\underline{E\,v\,d\,\omega}$ and its moment is $E\,\underline{v^2\,d\,\omega}$ as for the repulsive forces. Therefore for all the forces that section *aa′* exerts on the considered piece, we have the sum of their moments with respect to *o*, or the moment of elasticity

$$\sum M\{A\}_o = \frac{E}{\rho}\sum v^2\,d\,\omega \qquad \text{or} \qquad \frac{E\,I}{\rho},$$

the sum $\sum v^2\,d\,\omega$ called the moment of inertia being taken over the whole extent of section *aa′*.

101. *4th expression* of the moment of elasticity. In the applications which will be indicated, the radii of curvature of the pieces are always supposed very large in comparison with the lengths of these pieces, because it is important in constructions that deformations be hardly noticeable. If therefore we suppose an *x-axis* tangent at one of the points of the central line, passing through the centers of gravity of the cross-sections of the piece, the inclination of the curve on this axis will be everywhere very small.

By taking a *y-axis* in the plane of the curve and perpendicular to the *x-axis*, and supposing that the equation of the curve is *y = f(x)*, we know that generally the radius of curvature is

$$\rho = \frac{\left(1+(f'(x))^2\right)^{\frac{3}{2}}}{f''(x)}$$

but since in applications, the inclination $f'(x)$ is always small, we will only commit a very small error and obtain a great simplification if we write

$$\rho = \frac{1}{f''(x)} \quad \text{hence} \quad \sum M\{A\}_o = E\,f''(x)\sum v^2\,d\,\omega, \quad \text{or} \quad E\,I\,f''(x).$$

§:3 *Formulas for the moments of inertia of various plane surfaces and for the limits of the corresponding moments of elasticity.*

102. Parallelograms whose one side is parallel to the axis of the moment of inertia (fig. 17)

$$I = \int_{-\frac{1}{2}c}^{\frac{1}{2}c} v^2\, b\, d\, v = \frac{1}{2}\, b\, v^3 + c = \frac{1}{12}\, b\, c^3 = \Omega\, \frac{c^2}{12}$$

denoting by ω the area of the parallelogram.

The limit of the moment of elasticity is obtained by setting $v' = \frac{1}{2}c$ in the last formula of No. 99

$$M_1 = R_1 . \frac{bc^2}{6} = R_1\, \Omega\, \frac{c}{6}$$

103. Hollowed-out rectangle (fig. 18 and 19)

The moment of inertia is in this case the difference of two others which are calculated by the preceding formula. We have

$$I = \frac{1}{12}\left(b\, c^3 - b'\, c'^3\right) \qquad \text{and} \qquad M_1 = \frac{RI}{\frac{1}{2}c} = R_1\, \frac{b\, c^3 - b'\, c'^3}{6\, c}$$

In the application of these formulas and those which will be obtained in the following numbers, it must not be forgotten that they are founded on an hypothesis (93) which is only completely admissible for solid prisms, whose section differs little from a regular polygon. In the case of fig. 18 and 19 where the section presents two horizontal rectangles united by two vertical rectangles, if the latter became very thin, the first two would bend almost as if they were independent and subjected, each, to half the forces applied to the whole.

To avoid this effect, care is taken to fill the re-entrant angles with connecting curves called fillets.

104. Parallelogram whose diagonal is the axis of the moment of inertia (fig. 20)

$$I = \int_{-h}^{h} v^2 \left(b - \frac{bv}{h}\right) dv = b\left(\frac{v^3}{3} - \frac{v^4}{4\, h}\right) + c = \frac{1}{6}\, b\, h^3 = \Omega\, \frac{h^2}{6}$$

$$M_1 = \frac{R_1\, I}{h} = R_1\, \frac{6\, h^2}{6} = R_1\, \Omega\, \frac{h}{6}$$

105. Triangle whose one vertex is on the axis of the moment of inertia (fig. 21)

Same formula as in the previous number.

106. Triangle whose one side is parallel to the axis of the moment of inertia (fig. 22)

If the axis of the moment of inertia passed through the vertex, we would have

$$\sum y^2\, d\, \omega = \int_{o}^{h} y^2 . \frac{by\, dy}{h} = \frac{1}{4}\, b\, h^3$$

The axis passing through the center of gravity, we have, according to a known property of moments of inertia (note of No. 98)

$$I = \frac{1}{4} b h^3 - \frac{b h}{2} \cdot \left(\frac{2}{3} h \right)^2 = \frac{1}{36} b h^3 = \Omega \frac{h^2}{18}$$

As for M_1, there may be a distinction to be made (99) according to whether the lengthening occurs at the base b or at the opposite vertex. Let R_t be the value of R_1 for tension and R_p its value for compression. If the extension occurs at the base, M_1 will be the smaller of the two values

$$\Omega R_t \frac{h}{6} \quad , \quad \Omega R_p \frac{h}{12} ;$$

in the opposite case it will be the smaller of the two values

$$\Omega R_t \frac{h}{12} \quad , \quad \Omega R_p \frac{h}{6} ;$$

if, for example, $R_p = 3 R_t$, which is approximately the case for cast iron (99), M_1 will be $\Omega R_t \cdot \frac{h}{6}$ in the first case and $\Omega R_t \cdot \frac{h}{12}$ in the 2nd.

107. Rectangle whose sides are b and c, and whose side c makes an angle α with the axis of the moment of inertia (fig. 23)

Consider first the trapezoid $BCDE$, half of the rectangle. Let T and T' be the areas of triangles BCF, EDF. The moment of inertia of the trapezoid is (104)

$$\frac{1}{6} \left(T h^2 - T' h'^2 \right)$$

now $\quad T : T' :: h^2 : h'^2 \quad$ hence $\quad T b'^2 = T' b^2$;

$$\text{therefore } \frac{1}{6} \left(T h^2 - T' h'^2 \right) = \frac{1}{6} \left(T h^2 - T' h^2 + T h'^2 - T' h'^2 \right) = \frac{1}{6} \left(T - T' \right) \left(h^2 + h'^2 \right)$$

The figure shows that we have

$$\left. \begin{aligned} h &= \frac{1}{2} b \cos \alpha + \frac{1}{2} \sin \alpha \\ h' &= \frac{1}{2} b \cos \alpha - \frac{1}{2} \sin \alpha \end{aligned} \right\} \quad \text{hence} \quad h^2 + h'^2 = \frac{1}{2} \left(b^2 \cos^2 \alpha + c^2 \sin^2 \alpha \right)$$

Thus the moment of inertia of trapezoid $BCDE$ with respect to the axis oo is

$$\frac{1}{12} \left(T - T' \right) \left(b^2 \cos^2 \alpha + c^2 \sin^2 \alpha \right)$$

of which that of the rectangle, which is double, is, by designating by Ω the area of the rectangle, double of $T - T'$;

$$I = \frac{\Omega}{12} \left(b^2 \cos^2 \alpha + c^2 \sin^2 \alpha \right) \qquad \text{or} \qquad \frac{1}{12} b c \left(b^2 \cos^2 \alpha + c^2 \sin^2 \alpha \right)$$

$$M_1 = \frac{R_1 I}{h} = \Omega R_1 \frac{b^2 \cos^2 \alpha + c^2 \sin^2 \alpha}{6(\cos \alpha + \sin \alpha)}$$

These formulas coincide with those of No. 102 when $\alpha = 90°$.

108. When the preceding rectangle becomes a square, we have $b = c$ and

$$I = \frac{1}{12} A b^2$$

independent of α. But it is not the same for M_1 whose maximum, corresponding to $h = \frac{1}{2} b$ or $\alpha = 0$ is $\frac{1}{6} \Omega R_1 b$ and the minimum corresponding to $h = \frac{1}{2}\sqrt{2}$ or $\alpha = 45°$, is $\frac{1}{6\sqrt{2}} \Omega R_1 b$.

109. Cylindrical pipe. (fig. 24)

Suppose it first very thin: we will have

$$\sum v^2 d\omega = \sum u^2 d\omega \qquad \text{hence} \qquad \sum v^2 d\omega = \frac{1}{2} \sum (v^2 + u^2) d\omega$$

$$\text{so} \qquad I = \sum v^2 d\omega = \frac{I^2 \Omega}{2} \qquad \text{or} \qquad \pi r^3 \epsilon$$

$$\text{and} \qquad M = \frac{R_1 I}{r} = \frac{1}{2} R_1 \Omega r \qquad \text{or} \qquad R_1 \pi r^2 \epsilon \ (*)$$

calling r the radius and ε the small thickness of the pipe, subject to the observation of N° 103.

If the thickness is not very small compared to the radius, let r' and r'' be the radii of the inner and outer circumferences; we have

$$I = \int_{r'}^{r''} \pi r^3 dr = \frac{1}{4} \pi (r''^4 - r'^4) = \frac{\Omega}{4} (r''^2 + r'^2)$$

$$\text{and} \qquad M = \frac{R_1 I}{r''} = \frac{1}{4} R_1 \Omega \frac{r''^2 + r'^2}{r''}$$

110. Solid cylinder of revolution. Setting $r' = 0$, and $r'' = r$, we have

$$I = \frac{1}{4} \pi r^4 = \frac{\Omega}{4} r^2 \qquad \text{and} \qquad M' = \frac{1}{4} R_1 \Omega r$$

111. Solid ellipse, one of whose principal diameters is the axis of the moment of inertia (fig. 25)

Let $AB = 2a$, and $CD = 2b$. Comparing the ellipse to a circle whose diameter would be 2b, we see that the two surfaces can be decomposed into the same number of small rectangles of the same heights, two by two, and whose bases are in the ratio of a to b. Therefore (102) it is the same for the moments of inertia. We thus have (110)

$$I = \frac{1}{4} \pi b^4 . \frac{a}{b} = \frac{1}{4} \pi a b^3 \qquad \text{or} \qquad I = \Omega \frac{b^2}{4}$$

(*) Formulation may be unclear for "*or* $\pi r^3 \epsilon$" and "*or* $R_1 \pi r^2 \epsilon$", if one does not place oneself back in the context of an oral lecture, allowing elision, respectively of *I*= and *M*=

and $$M_1 = R_1 \, \Omega \, \frac{b}{4}$$

If the axis of moments were any diameter, calling it $2a'$, designating by $2b'$ its conjugate and by α their angle, it would suffice in the preceding formulas to replace a by a' and b by $b' \sin \alpha$.

112. Any plane surface (fig. 26). Let a surface *KLNM* bounded by two straight lines *KL*, *MN,* parallel to the axis oo of the moment of inertia, and by two curves *KM*, *LN*. Let u denote the length of a chord PQ parallel to *oo*, and v its distance from this axis.

For this surface, the moment of inertia is

$$I = \int_K^{K+l} v^2 \, u dv \, .$$

To obtain the approximate value of this integral, we divide the distance from *KL* to *MN* into an even number n of equal parts, and drawing through the division points parallels to the axis which, including *KL* and *MN,* will be denoted by $u_0, u_1, u_2, \ldots u_n$, we denote by δ the distance $\frac{l}{n}$ of two consecutive parallels, and set according to Th. Simpson's formula

$$I = \frac{\delta}{3}\left(K^2 u_0 + 4\,(K+\delta)^2 u_1 + 2\,(K+2\,\delta)^2 u_2 + 4\,(K+3\,\delta)^2 u_3 + \ldots + 4\,(K+l-\delta)^2 u_{n-1} + (K+l)^2 u_n\right)$$

When the curves *KM*, *LN* are not very irregular, it suffices to take $n = 4$. The parallels u_0, u_n may be zero.

§: 4 Prisms subjected transversely to a uniformly distributed load and to discontinuous forces in the same plane.

1° Generalities on this subject

113. Let *OB* (fig. 27) be a portion of a prism, whose longitudinal edges make, at the end *O*, a very small angle *COD* or α with the horizontal *OC*.

The length of the prism is denoted by a. Its uniformly distributed load is pa. In addition, this body is subjected to vertical forces $P', P'', \ldots, P^{(n)}$ whose distances from the origin *o* are $x', x'', \ldots, x^{(n)}$.

Given the slight curvature of the piece, these distances are indifferently $OM', OM'', \ldots, OM^{(n)}$, measured along the piece, or $ON', ON'', \ldots, ON^{(n)}$ measured along the horizontal O_x.

Let M be any point on the mean fiber of the piece, between O and M', and let x and y be its rectangular coordinates. Applying to the solid comprised between the plane aMa' and the end B the general considerations of §: 2, Nos. 93 and seq., we will have (95)

$$[1] \ldots \sum M\{A\}_M = \frac{1}{2} p\,(a-x)^2 + P'(x'-x) + P''(x''-x) + \ldots P^{(n)}(x^{(n)}-x)$$

Putting for $\sum M\{A\}_M$ either its second expression (99), or the third (100), we have two equations

$$\left.\begin{array}{l} [\,2\,]\ldots\ldots\ldots \dfrac{R}{V'}I \\ [\,3\,]\ldots\ldots\ldots \dfrac{E}{\rho} \end{array}\right\} = \frac{1}{2}p\,(a-x)^2 + P'(x'x) + P''(x''-x) + \ldots\ P^{(n)}(x^{(n)}-x)$$

In these equations, v' and I are constants which depend on the section of the prism, E is a constant depending on the substance of the prism, R' (maximum resistance per unit area in section aa') and ρ (radius of curvature at M) are variables depending on x, the sole variable of the second member.

For any point situated between M' and M'', we will have two equations which will differ from the two preceding ones only in that P' will disappear. And so on up to the end B of the piece.

114. If the forces $P', P'' \ldots., P^{(n)}$ are positive, i.e., if they have the direction indicated in the figure, the second member common to the two above equations decreases with x: the greatest value of R' and the greatest flexure (or the smallest radius of curvature) occur at the origin o.

This may no longer be the case when the forces P are all or partially negative, and the discussion of the values of the second member will make known the points of greatest tension or greatest flexure, and the points of inflection (i.e., of no flexure) where the tension is zero as at end B.

If the algebraic sum of the forces $P', P'', \ldots., P^{(n)}$ were zero, they would be equivalent to a couple whose moment with respect to M would be constant; and if at the same time pa were zero, ρ would be constant between O and M', i.e., the curve OM' would be a circle.

115. If in equation [1] we put for $\sum M\{A\}_M$ its fourth expression (101), we have

$$[\,4\,]\ldots\ldots\ EI\int''(x) = \frac{1}{2}p\,(a-x)^2 + P'(x'-x) + P''(x''-x) + \ldots. + P^{(n)}(x^{(n)}-x)$$

equation which only holds between O and M'. Multiplying by dx and integrating, we have, for any point comprised between O and M',

$$[\,5\,]\ldots\ EI\left(f'(x) - \tan\alpha\right) = \frac{1}{2}p\left(a^2x - ax^2 + \frac{x^3}{3}\right) + P'\left(x'x - \frac{x^2}{2}\right) + P''\left(x''-x\right) + \ldots. + P^{(n)}\left(x^{(n)}-x\right)$$

Multiplying again by dx and integrating, we have, still for a point between O and M',

the origin of coordinates being at O,

$$[\,6\,]\quad EI\,(x - y\,\tan\alpha) = \frac{1}{2}p\left(\frac{a^2x^2}{2} - \frac{ax^3}{3} + \frac{x^4}{12}\right) + P'\left(\frac{x'x^2}{2} - \frac{x^3}{6}\right) + P''\left(\frac{x''x^2}{2} - \frac{x^3}{6}\right) + P^{(n)}\left(\frac{x^{(n)}x^2}{2} - \frac{x^3}{6}\right)$$

Thus the quantities α, P', x', P'', x'', $\ldots.,P^{(n)}$, $x^{(n)}$ being supposed given, we would find the inclination $f'(x)$ and the ordinate at any point of the curve OM', whose abscissa x is given.

Now between M' and M'', we have an equation similar to formula [4] except for the disappearance of P', namely

[7] . . . . . $EI f''_1(x) = \frac{1}{2} P (a-x)^2 + P''(x''-x) + ... + P^{(n)} (x^{(n)} - x)$;

its integral, analogous to formula [5] except for the constant, is

$$EI f'_1(x) - C = \frac{1}{2} p \left(a^2 x - a x^2 + \frac{x^3}{3} \right) + P'' \left(x'' - x - \frac{x^2}{2} \right) + ... + P^{(n)} \left(x^{(n)} x - \frac{x^2}{2} \right)$$

If in this equation and in [5] we set $x = x'$, the inclinations $f'(x)$ and $f'_1(x)$ both belonging to point M' become equal, and we have by subtraction

$$C - EI \tan \alpha = P' . \frac{x'^2}{2} ;$$

The preceding equation therefore becomes

[8] $EI \left(f'_1 x - \tan \alpha \right) = \frac{1}{2} p \left(a^2 x - a x^2 + \frac{x^3}{3} \right) + P' \frac{x'^2}{2} + P'' \left(x'' x - \frac{x^2}{2} \right) + ... + P^{(n)} \left(\frac{x^{(n)} x^2}{2} - \frac{x^3}{6} \right)$

Its integral is, for any point from M' to M''

$$EI (y - x \tan \alpha) - C' = \frac{1}{2} p \left(\frac{a^2 x^2}{2} - \frac{a x^3}{3} + \frac{x^4}{12} \right) + P' \frac{x'^2 x}{2} + P'' \left(\frac{x'' x^2}{2} - \frac{x^3}{6} \right) + ... + P^{(n)} \left(\frac{x^{(n)} x^2}{2} - \frac{x^3}{6} \right) .$$

If in this equation and in [6] we set x and y equal to the coordinates x' and y' of point M', these two equations are satisfied and subtraction gives the constant

$$C' = - P' \frac{x'^3}{6} .;$$

The equation of the curve between M' and M'' is therefore

[9] $EI (y - x \tan \alpha) = \frac{1}{2} p \left(\frac{a^2 x^2}{2} - \frac{a x^3}{3} + \frac{x^4}{12} \right) - P' \frac{x'^3}{6} + P' \frac{x'^3 x}{2} + P'' \left(\frac{x'' x^2}{2} - \frac{x^3}{6} \right) + ... + P^{(n)} \left(\frac{x^{(n)} x^2}{2} - \frac{x^3}{6} \right)$

Proceeding in this way we will obtain the inclination of the curve and the ordinate at end B.

We will apply these general results to some particular cases.

2° Portions of horizontal prism with one end subjected to a vertical force at the other end and additionally loaded uniformly over its length.

We must set in the preceding formulas

$\alpha = 0$, $x' = a$, $P' = P$ (extreme force positive or negative)

$P' = P'' = ... = 0$. p remains the uniform load per unit length; and the formulas become, as one could otherwise obtain them directly,

$$\left.\begin{array}{ll}[\,1,\,2°\,] & \sum M\{A\}_M \ldots\ldots \\ [\,2,\,2°\,] & = \dfrac{R'I}{v'} \ldots\ldots\ldots \\ [\,3,\,2°\,] & = \dfrac{EI}{\rho} \ldots\ldots\ldots\ldots \\ [\,4,\,2°\,] & = EI\,f''(x) \ldots \end{array}\right\} = \frac{1}{2}\,p\,(\alpha - x)^2 + P\,(a - x)$$

$$[\,5,\,2°\,] \qquad EI\,f'(x) = \frac{1}{2}\,p\left(\alpha^2 x - ax^3 + \frac{x^3}{3}\right) + P\left(ax - \frac{x^2}{2}\right)$$

$$[\,6,\,2°\,] \qquad EI\,y = \frac{1}{2}\,p\left(\frac{a^2 x^2}{2} - \frac{ax^3}{3} + \frac{x^4}{12}\right) + P\left(\frac{ax^2}{2} - \frac{x^3}{6}\right)$$

117. If P is positive, the greatest value of the second member of the first equations corresponds to $x = 0$; thus calling R_1 the greatest resistance per unit area of the substance of the prism (99), the forces pa and P, one distributed, the other applied at B, must satisfy the equation

$$\frac{R'I}{v'} = \frac{1}{2}\,pa\,.\,a + Pa$$

which proves that, as regards the state of tension and flexure of the piece at O, and consequently as regards the danger of reaching and exceeding the elastic limit, the distributed load pa produces the same effect as if it were applied at the middle of the piece, or as if a force half $\frac{pa}{2}$ were applied at end B.

118. If P is negative, designating by the letter P only the absolute value of the extreme force, equation [2] becomes

$$\frac{R'I}{v'} = \frac{1}{2}\,p\,(a - x)^2 - P\,(a - x)$$

or

$$\frac{RI}{v'} = \frac{1}{2}\,p\left(a - x - \frac{P}{p}\right)^2 - \frac{1}{2}\,\frac{P^2}{p}$$

The second member can be represented by the ordinate of a parabola, whose algebraic minimum is $\frac{1}{2}\,\frac{P^2}{p}$, corresponding to $x = a - \frac{P}{p}$.

At the point having this abscissa, the second member is negative, i.e., the curve is concave upwards. The same holds at all points situated between end B where $x = a$ makes the second member zero and the other point where this second member becomes zero again, which corresponds to $x = a - \frac{2P}{p}$. From this point, when this last quantity is positive, i.e., when $pa > 2P$, as one moves back towards the origin o of x, the second member, having become positive, increases and takes for $x = 0$ the value $\frac{1}{2}\,pa^2 - Pa$.

Thus, to determine the point of greatest curvature or greatest tension, several cases must be distinguished:

1° If $a > \frac{P}{p}$ but $a < \frac{2P}{p}$, the curvature is also everywhere negative but the maximum absolute value of the moment of elasticity, corresponding to $x = a - \frac{P}{p'}$, is $\frac{1}{2}\frac{P}{p}$.

2° If $a < \frac{P}{p}$ or $pa < P$, the curvature is sometimes negative, and its maximum absolute value of the moment of elasticity, corresponding to $x = 0$, is $Pa - \frac{1}{2}pa^2$.

3° If $a > \frac{2P}{p}$, the curvature is zero at the point whose abscissa is $x = a - \frac{2P}{p}$, and the maximum absolute value of the moment of elasticity is the greater of the two quantities

$$\frac{1}{2}\frac{P^2}{p} \qquad \text{and} \qquad \frac{1}{2}pa^2 - Pa\ .$$

The maximum moment of elasticity being thus determined, we can verify whether it does not exceed the value $\frac{R_i I}{v'}$ . . . . (99)

119. Setting $x = a$ in equation [6] of No. 116, we will obtain for the value of y the height h of the extreme point B below the horizontal 0_x, namely:

$$h = \frac{1}{E\,I}\left(\frac{pa^4}{8} + \frac{Pa^3}{3}\right).$$

Thus the part of h due to the uniformly distributed load p is only $\frac{3}{8}$ of what it would be if this same load were concentrated at end B.

3° Prism placed on two level supports, and subjected to a vertical force P at a point C, besides a load pa *uniformly distributed over its length* a *- figure 28*

120. The reactions exerted by the supports A, B can be supposed vertical. Their values are

$$\text{at } A \ldots \frac{1}{2}pa + P\frac{d'}{a}, \qquad \text{at } B \ldots \frac{1}{2}pa + P\frac{d}{a}.$$

For a point situated between A and C, calling x its distance from end A, we have the variable moment of elasticity equal to the moment with respect to this point of the forces acting between this same point and A:

$$[\,1\,] \ldots \frac{R'I}{V'} = \left(\frac{1}{2}pa + \frac{Pd'}{a}\right)x - \frac{1}{2}px^2 = \frac{1}{2}px\left(a + \frac{2Pd'}{pa} - x\right)$$

This quantity is proportional to the square of the ordinate corresponding to the abscissa x in a circle whose diameter equal to $a + \frac{2Pd'}{pa}$ would have its origin at A. Therefore if $d < \frac{1}{2}a$, the greatest value of R' on the curve AC occurs at point C. If on the contrary $d > \frac{1}{2}a$, the maximum of R' corresponds to that of the ordinate of the circle whose diameter AD (fig. 29) is obtained by extending to the right of end B the length $\frac{2Pd'}{pa}$. The maximum corresponding to point O, midpoint of AD, we see that if this midpoint is between A and C, that is where the greatest tension R' is; in the opposite case it is at C.

It therefore suffices to verify whether $\frac{1}{2}AD$ is smaller or larger than AC, i.e., whether

$\frac{1}{2}a+\frac{Pd'}{pa}$ is < or > d.

In the first case, we have the maximum of $\frac{R'I}{v'}$ by setting in equation [1],

$$x=\frac{1}{2}\quad AD=\frac{1}{2}a+\frac{Pd'}{pa}$$

which gives

$$\frac{R'I}{v'}=\frac{1}{2}p\left(\frac{1}{2}a+\frac{Pd'}{pa}\right)^2$$

In the second case, the maximum of $\frac{R'I}{v'}$ is obtained by setting $x = d'$ in equation [1] and we have

$$\frac{R'I}{v'}=\left(P+\frac{1}{2}pa\right)\frac{dd'}{a}$$

as if the uniformly distributed load pa were replaced by a single force $\frac{1}{2}pa$ combined with the force P applied at C.

By means of these formulas, knowing a, d, pa, I and v', we can, putting for R' the value R_1 which must not be exceeded, determine the greatest value that can be given to P.

121. When C is the midpoint of AB, we have $d=d'=\frac{1}{2}a$, we are in the second case and the above formula becomes

$$\frac{R'I}{v'}=\frac{a}{4}\left(P=\frac{1}{2}pa\right)$$

Therefore, as regards the tension or pressure R' and the curvature at C, the distributed load is equivalent to a force $\frac{1}{2}pa$ applied at C.

122. In the particular case where pa is zero or negligible compared to P, equation [1] of No. 120 reduces to

$$\frac{R'I}{v'}=\frac{Pd'}{a}x$$

which it would be easy to set *a priori*, proves that the resistance R', zero at each end of the piece, increases up to C, the point of application of the force P. Its maximum, consequently occurring at C, satisfies the equation

$$\frac{R'I}{v'}=\frac{Pd'd}{a}$$

and the greatest value of P, so as not to exceed the resistance R_1, is

$$P=\frac{a}{dd'}\frac{v'}{R_1 I}$$

123. The particular case where P is zero is also included in the preceding formulas.

124. In the general hypothesis of No. 120, one could propose to determine the curve of the mean fiber of the piece. One would proceed according to the procedure indicated in No. 115:

assuming the inclination $tan\ \alpha$ of this curve at point A known, one would determine the ordinate and the inclination of the curve at C as a function of $tan\ \alpha$, then the ordinate of point B, and by equating this last to zero, one would have an equation to determine $tan\ \alpha$, and consequently the ordinate for any abscissa of the curve.

125. In the particular case where C is the midpoint of AB, the curve is horizontal at this point, and we obtain the deflection by substituting in the expression for h of No. 119 $\frac{1}{2}a$ for a, and $\frac{1}{2}(P+pa)$ for P, then changing the sign of the result. We thus have

$$f=\frac{1}{48}\frac{a^3}{EI}\left(P+\frac{5}{8}pa\right)$$

therefore, as regards the production of deflection, the uniformly distributed load pa is equivalent to a force $\frac{5}{8}pa$ a applied at the midpoint C.

4° Prism placed on two level supports AB, and subjected to discontinuous vertical forces P′, P″, P‴ .....

126. Let (fig. 30) $AB = a$, $AC' = x'$, $AC'' = x''$ . . . . . The reactions Q', Q of the supports being sensibly vertical are determined by two of the three equations

$$Q\,a=P'\,x'+P''\,x''+P'''\,x'''+.....$$
$$Q'\,a=P'\,(a-x')+P''\,(a-x'')+P'''\,(a-x''')+....$$
$$Q+Q'=P'+P''+P'''+.....$$

That established, one can apply what was said in Nos. 113, 114 and 115, with this difference that the equations will be simpler, because there is no longer a continuously distributed force pa.

Still denoting by R' and ρ two variables which are the extreme resistance and the radius of curvature in the section determined by the abscissa x.

We have for any point

on the curve AC, $\quad\frac{R'\,I}{v'}=\frac{EI}{\rho}=-Q'\,x$

on the curve $C'C''$, $\quad\frac{R'\,I}{v'}=\frac{EI}{\rho}=-Q'\,x+P'\,(x-x')$

on the curve $C^{(n-1)}C^{(n)}$, $\quad\frac{R'\,I}{v'}=\frac{EI}{\rho}=-Q\,(a-x)+P^{(n)}\,(x^{(n)}-x)$

on the curve $C^{(n)}B$, $\quad\frac{R'\,I}{v'}=\frac{EI}{\rho}=-Q\,(a-x)$

We will note

1° that in the calculation of the sum of moments which provides the second member of one of these equations, it is indifferent to take the forces situated to the right or those to the left of the considered portion.

2° that when the second member is negative, (which depends on the values and signs of the forces *P,* and which is always the case when these forces are positive) the curve is concave towards the negative sense of these forces.

3° that the two curves which meet at a point *C* of application have at this point not only the same tangent but the same curvature.

4° that the second member of each equation being of the first degree in *x*, its greatest value over the extent of the part of the curve to which it belongs occurs at one of the extreme points of this part; from which it follows that if we set $x = x'$ in the 1st equation, $x = x''$ in the 3rd, the greatest of the values obtained for the second members, being equal to $\frac{EI}{\rho}$ and to $\frac{R'I}{v'}$ will give the greatest curvature $\frac{1}{\rho}$ and the greatest resistance R' over the extent of the piece.

127. The procedure indicated in Nos. 124 and 125 would lead to the determination of the shape of the piece over its entire extent.

5° Prism held horizontal at a given point A, resting at one end B on a support, subjected in the interval to a load pa uniformly distributed over the length AB or a, and to a distinct force P applied at point C. (fig. 31)

128. Let $AC = x'$ and let h be the height of point B below the horizontal A_x, x' and h are given, as well as a, p and P.

Let Q be the sensibly vertical reaction of support B on the piece.

The method for determining this unknown force is based on this remark: that if it were given, one would determine by the analysis of No. 115 the ordinate h of point B. One would directly obtain the following six equations which are deduced from those of No. 115 by setting $\alpha = 0$, replacing P' by P, P'' by $-Q$, x'' by a, and suppressing the terms relating to forces P''' . . . . $P^{(n)}$. We thus have, between A and C,

$$[1] \ldots . \quad EI\, f''(x) = \frac{1}{2}\, p\,(a-x)^2 + P\,(x'-x) - Q\,(a-x)$$

$$[2] \ldots . \quad EI\, f'(x) = \frac{1}{2}\, p\left(a^2 x - a\,x^2 + \frac{x^3}{3}\right) + P\left(x'x - \frac{x^2}{2}\right) - Q\left(ax - \frac{x^2}{2}\right)$$

$$[3] \ldots . \quad EI\, y = \frac{1}{2}\, p\left(\frac{a^2 x^2}{2} - \frac{a\,x^3}{3} + \frac{x^4}{12}\right) + P\left(\frac{x'x^2}{2} - \frac{x^3}{6}\right) - Q\left(\frac{a\,x^2}{2} - \frac{x^3}{6}\right)$$

and between C and B

$$[4] \ldots . \quad EI\, f''_1(x) = \frac{1}{2}\, p\,(a-x)^2 - Q\,(a-x)$$

$$[5] \ldots . \quad EI\, f'_1(x) = \frac{1}{2}\, p\left(a^2 x - ax^2 + \frac{x^3}{3}\right) + P\left(\frac{x'^2}{2}\right) - Q\left(ax - \frac{x^2}{2}\right)$$

$$[\,6\,]\ldots.\quad EI\,y=\frac{1}{2}\,p\left(\frac{a^2\,x^2}{2}-\frac{ax^3}{3}+\frac{x^4}{12}\right)-P\,\frac{x'^3}{6}+P\,\frac{x'^2\,x}{2}-Q\left(\frac{ax^2}{2}-\frac{x^3}{6}\right).$$

Now, the value of Q must be such that this equation is satisfied by $x = a$ and $y = h$. We therefore have

$$EI\,h=\frac{1}{2}\,pa^4\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{12}\right)+\frac{Px'^2}{2}\left(a-\frac{x'}{3}\right)-Q\,\frac{a^3}{3}$$

hense
$$Q=P\,\frac{x'^2\,(3\,a-x')}{2\,a^3}+\frac{3}{8}\,pa-3\,\frac{EI\,h}{a^3}\quad\ldots\ldots\ldots[\,7\,]$$

This value being substituted in the six preceding equations, we will know for any point the moment of elasticity, the resistance R', the curvature $\frac{1}{\rho}$, the inclination $f'(x)$ or $f'_1(x)$, and the ordinate y.

129. If there exists a point of inflection between A and B, it will be determined by setting the second member of equation [1] or [4] to zero.

130. Particular case. $h = 0$ and $p = 0$, from which results $Q=P\,\frac{x^2(3\,a-x)}{2\,a^3}$.

Equation [4] becomes $M\,\{A\}=-Q\,(a-x)$ which shows that from C to B the absolute value of curvature goes on decreasing and becomes zero at B.

Equation [1] can be written thus

$$M\,\{A\}=Px'-Qa-(P-Q)\,x\quad\ldots\ldots[\,8\,]$$

a quantity positive or negative depending on whether the concavity of the curve is downwards or upwards.

This function being of the first degree, all its values lie between the extremes, namely

for $x=0$ $\qquad Px'-Qx$

for $x=x'$ $\qquad -Q\,(a-x')$.

The first is negative evidently. The 2nd is positive, otherwise all values of $M\,\{A\}$ from A to C would be negative, and as it is the same between C and B, the curve, everywhere concave upwards, could not pass through B.

That established, the greatest curvature will occur at A or at C depending on whether the positive quantity

$$Px'-Qa \quad \text{is} \quad > \quad \text{or} \quad < \quad Q\,(a-x').$$

Let us set $\qquad Px'-Qa-Q\,(a-x')=\beta\,P\,x'$

the two inequality cases will occur according to whether β is positive or negative. Substituting the value of Q, we have

$$P\,x'-P\,\frac{x'^2\,(3\,a-x')}{2\,a^3}\,(2\,a-x')=\beta\,P\,x'$$

or $\quad 2-\frac{x'}{a}\left(3-\frac{x'}{a}\right)\left(2-\frac{x'}{a}\right)=\beta\quad$ or finally $\quad n^3-5\,n^2+6\,n-2=-\beta$

setting $\frac{x}{a} = n$, if we suppose $\beta = 0$, one of the roots of this equation is $n = 1$, which had to be since if $x' = a$ the curvatures at A and C are both zero.

The last equation divided by $(n - 1)$ becomes

$$n^2 - 4\,n - 2 = \frac{\beta}{1-n} \quad \text{hence} \quad n = 2 - \sqrt{2 + \frac{\beta}{1-n}},$$

only one value, since $n < 1$.

If $\beta = 0$ we have $n = 0.586$ and the two curvatures of opposite signs at A and C have the same absolute magnitude.

If $\beta > 0$ we have $n < 0.586$; it is the opposite if $\beta < 0$.

Thus according to whether x' is $>$ or $<$ $0.586\,a$, the greatest curvature is at A or C.

131. *Deflection length*. Supposing that the lowest point of the curve is between C and B, set the second member of equation [5] to zero, with $p = 0$ and $Q = P\,\frac{x'^2(3a - x')}{2a^3}$.

we will find $$x = a\left(1 - \sqrt{\frac{a - x'}{3\,a - x'}}\right)$$

This quantity must be $> x'$, we will put $a - a\sqrt{\frac{a-x'}{3a-x'}} > x'$

hence $a - x' > a\sqrt{\frac{a-x'}{3a-x'}}$ or $(3\,a - x')(a - x') > a^2$, or $4\,a^2 - 4\,ax' + x'^2 > 2\,a^2$, or $2\,a - x' > a\sqrt{2}$, finally $x' < a\,(2 - \sqrt{2})$ or $x' < 0{,}586\,a$

Thus according as x' is $<$ or $>$ 0.586 a, the lowest point of the curve is between C and B or between C and A.

Assuming the first hypothesis, we will substitute the above value of x in equation [6] and we will have, denoting the deflection by f

$$EI\,f = P\,\frac{x^2(a - x')}{6}\sqrt{\frac{a - x'}{3\,a - x'}}$$

132. <u>More particular case</u> $\rho_0 = 0$, $p = 0$, $x' = \frac{1}{2}a$

The greatest curvature is at A, and the lowest point is between B and C. Substituting the above values in equation [7] of No. 128 we have

$$Q = \frac{5}{16}P$$

putting this value in [8] of No. 130, and setting $x = 0$, we have at point A

$$\frac{EI}{\rho} = \frac{R'I}{v'} = \frac{1}{2}Pa - \frac{5}{16}Pa = \frac{3}{16}Pa$$

while if the piece were freely supported at A (121) as at B, the maximum of $\frac{EI}{\rho}$ would be at C and equal to $\frac{1}{4}Pa$.

Substitution of $x' = \frac{1}{2}a$ in the final formula of No. 131 gives

$$f = \frac{1}{48}\frac{Pa^3}{EI}\sqrt{\frac{1}{5}} = \frac{0{,}45}{48}\frac{Pa^3}{EI}$$

while if the piece were freely supported at both ends, the deflection would be (125) $\frac{1}{48}\frac{Pa^3}{EI}$.

133. Other particular cases: $h = 0$, and $P = 0$.

The value of Q [7] becomes $Q = \frac{3}{8}pa$ and equations [1] and [4] of No. 128 reduce, for the whole curve AB, to

$$M\{A\} = \frac{1}{2}p(a-x)^2 - \frac{3}{8}pa(a-x) = \frac{1}{2}p(a-x)(\frac{1}{4}a - x)$$

this quantity is zero at $x = a$, i.e., at end B, and at $x = \frac{1}{4}a$, which gives the inflection point of the piece. The maximum curvature between the inflection point and end B is obtained by setting the derivative of the second member to zero, namely

$$\frac{-1}{4}a + x - a + x = 0 \qquad \text{or} \qquad x = \frac{5}{8}a\ .$$

Substituting we have the sought maximum of $M\{A\}$ in absolute value

$$\frac{EI}{\rho} = \frac{T'I}{v'} = \frac{9}{128}pa^2\ .$$

From end A up to the inflection point, the curvature decreases; the greatest value of the moment of elasticity obtained by setting $x = 0$ in the general expression above is

$$\frac{EI}{\rho} = \frac{T'I}{v'} = \frac{1}{8}pa^2 = \frac{16}{128}pa^2$$

and consequently exceeds the above maximum.

When the piece is simply supported at both ends, we also have at the point of greatest curvature $\frac{EI}{\rho} = \frac{1}{8}pa^2$, but this point is at the middle of the piece.

134. The lowest point of the curve will be found by setting the second member of equation [5] to zero after having set $p = 0$ and $Q = \frac{3}{8}pa$. We have, by removing the x factor which responds to point A,

$$\frac{1}{2}p\left(a^2 - ax = \frac{x^2}{3}\right) - \frac{3}{8}pa\left(a - \frac{x}{2}\right) = 0 \qquad \text{or} \qquad x^2 - \frac{15}{8}ax + \frac{3}{4}a^2 = 0$$

hence $$x = 0{,}58\ \ldots\ a$$

Substituting this value in equation [6], with $P = 0$ and $Q = \frac{3}{8}pa$, we find for the deflection f

$$EI\,f = 0{,}0067\ldots . pa\,.\,a^3$$

approximately double the preceding value.

6° Prism held horizontal at two points A and B at the same level, subjected in the interval to a load pa uniformly distributed over the length AB or a and a distinct force P applied at point C

135. (fig. 32) Let Q be the vertical reaction of the support at point B. One can always suppose that the piece is held horizontal at B by means of an upper support which exerts at B' a vertical reaction Q' whose distance from point A is separated by a'.

Considering Q' and Q as known, we will follow a procedure analogous to that of No. 128; we will find the inclination and the ordinate at point B, and equate both to zero, which will determine Q and Q'.

To this end, we set $\alpha = 0$ in the six formulas of No. 115, replace P' by P, x' by δ, P'' by $-Q$, x'' by a, P''' by Q', x''' by a', suppressing the terms relating to other forces $P^{IV} \ldots . P^{(n)}$.

We thus have the following six equations, which could moreover be obtained directly;

1° for the curve AC

$$[\,1\,] \ldots . EI\, f''(x) = \frac{1}{2}\, p\,(a-x)^2 + P\,(\delta - x) - Q\,(a-x) + Q'\,(a'-x)$$

$$[\,2\,] \ldots . EI\, f'(x) = \frac{1}{2}\, p\left(a^2\, x - a x^2 + \frac{x^3}{3}\right) + P\left(\delta\, x - \frac{x^2}{2}\right) - Q\left(a x - \frac{x^2}{2}\right) + Q'\left(a'\, x - \frac{x^2}{2}\right)$$

$$[\,3\,] \ldots . EI\, y = \frac{1}{2}\, p\left(\frac{a^2\, x^2}{2} - \frac{a x^3}{3} + \frac{x^4}{12}\right) + P\left(\frac{\delta\, x^2}{2} - \frac{x^3}{6}\right) - Q\left(\frac{a x^2}{2} - \frac{x^3}{6}\right) + Q'\left(\frac{a'\, x^2}{2} - \frac{x^3}{6}\right)$$

2° for the curve CB

$$[\,4\,] \ldots . EI\, f''_1(x) = \frac{1}{2}\, p\,(a-x)^2 - Q\,(a-x) + Q'\,(a'-x)$$

$$[\,5\,] \ldots . EI\, f'_1(x) = \frac{1}{2}\, p\left(a^2\, x - a x^2 + \frac{x^3}{3}\right) + P\,\frac{\delta^2}{2} - Q\left(a x - \frac{x^2}{2}\right) + Q'\left(a'\, x - \frac{x^2}{2}\right)$$

$$[\,6\,] \ldots . EI\, y = \frac{1}{2}\, p\left(\frac{a^2\, x^2}{2} - \frac{a x^3}{3} + \frac{x^4}{12}\right) - \frac{P\,\delta^3}{6} + \frac{P\,\delta^2\, x}{2} - Q\left(\frac{a x^2}{2} - \frac{x^3}{6}\right) + Q'\left(\frac{a'\, x^2}{2} - \frac{x^3}{6}\right)$$

That established, according to the statement, equations [5] and [6] must be satisfied when we set $x = a$, $f'_1(a) = 0$; $y = 0$, which gives

$$[\,7\,] \ldots . 0 = \frac{1}{6}\, p a^3 + P\,\frac{\delta^2}{2} - Q\,\frac{a^2}{2} + Q'\,a\left(a' - \frac{a}{2}\right)$$

$$[\,8\,] \ldots . 0 = \frac{1}{8}\, p a^4 + P\,\frac{\delta^2}{2}\left(a - \frac{\delta}{3}\right) - Q\,\frac{a^3}{3} + Q'\,\frac{a^2}{2}\left(a' - \frac{a}{3}\right)$$

two equations of the first degree for the two unknowns Q and Q'.

136. Particular case $p = 0$. To simplify the calculations, let us abstract from the weight of the piece and the distributed load. By $p = 0$, equation [1] can be written thus

$$\frac{EI}{\rho} = \frac{R'\,I}{v'} = P\,\delta + Q'\,a' - Q a - (P - Q + Q')\,x$$

this expression of the moment of elasticity being of the first degree, it follows that from A to C the greatest curvature or the greatest resistance R' occurs at A or at C. Let μ and μ' be the moments of elasticity at these two points; they are obtained by successively setting $x = 0$ and $x = \delta$. We thus have

$$\mu = P\,\delta + Q'\,a' - Q\,a \qquad \text{and} \qquad \mu' = Q'\,a' - Q\,a + (Q - Q')\,\delta$$

To compare these two quantities, we must derive the values of Q and Q' from equations [7] and [8] after having set $p = 0$. We obtain

$$Q' = P'\,\frac{\delta^2 (a - b)}{a^2 (a - a')} \qquad \text{and} \qquad Q = P\,\frac{\delta^2 (3\,aa' - 2\,a^2 + \delta\,a - 2\,\delta\,a')}{a^2 (a - a')}$$

from which one concludes

$$Q'\,a' - Q\,a = P\,\frac{\delta^2 (\delta - 2\,a)}{a^2} \quad \text{and} \quad Q - Q' = P\,\frac{\delta^2 (3\,a - 2\,\delta)}{a^3}$$

and substituting these values in the above expressions of μ and μ'

$$\mu = P\,\frac{\delta\,(a - d)^2}{a^2} \qquad \text{and} \qquad \mu' = -2\,P\,\frac{\delta^2 (a - \delta)^2}{a^2}$$

thus μ is positive, μ' negative, and the ratio of their absolute values

$$-\frac{\mu}{\mu'} = \frac{a}{2\,\delta};$$

so that μ is greater or smaller than μ' depending on whether δ is smaller or larger than $\frac{1}{2}a$, i.e., whether point C is closer to or farther from A than from B.

The piece being horizontal at B as at A, what has just been said about the curve AC applies exactly to the curve BC provided only that we replace the distance AC, which we denote δ, by the distance BC, which we will represent by δ'; so that if we call μ'' the moment of elasticity at point B, we will have

$$-\frac{\mu''}{\mu'} = \frac{a}{2\,\delta'} \quad \text{and consequently} \quad \frac{\mu}{\mu''} = \frac{\delta'}{\delta}.$$

It follows that if $\delta' > \delta$, consequently $a > 2\,\delta$ and $a < 2\,\delta'$, the order of magnitude of the moment of elasticity is $\mu > \mu' > \mu''$.

In short, if we call δ the smaller of the two distances from point C to points A and B, the greatest moment of elasticity over the whole extent of the piece is

$$P\,\frac{\delta\,(a - \delta)^2}{a^2}$$

137. <u>More particular cases</u>: $p = 0$ and $\delta = \frac{1}{2}a$. We then have

$$\mu = \mu' = \mu'' = \frac{1}{8}Pa$$

while if the piece were simply supported at A and B, the greatest moment of elasticity would be at C and (121) would be

$$\frac{1}{4} Pa$$

To obtain the value of the deflection f, in this case, it suffices to set in formula [3] of No. 135, $p = 0$, $x = \delta = \frac{1}{2} a$ and put in it the value that the forces Q and Q' take in these hypotheses according to the formulas of No. 136, namely

$$Q' = \frac{1}{8} P \frac{a}{a' - a} \qquad \text{and} \qquad Q = \frac{1}{8} P \cdot \frac{4\,a' - 3\,a}{a' - a};$$

we find

$$EI\,f = \frac{1}{24} P \left(\frac{a}{2}\right)^3$$

while if the piece were simply supported at A and B we would have

$$EI\,f = \frac{1}{6} P \left(\frac{a}{2}\right)^3.$$

Between A and C there is a point of inflection whose abscissa will be obtained by making the preceding substitutions in the second member of equation [1] and equating it to zero. We find $x = \frac{1}{4} a$, as was easy to foresee.

138. Other particular case: prism uniformly loaded, $P = 0$. By this hypothesis, equations [7] and [8] of No. 135 reduce to

$$0 = \frac{pa^2}{3} - Qa + Q'(2\,a' - a)$$

$$0 = \frac{3\,pa^2}{8} - Qa + Q' \left(\frac{3}{2} a' - \frac{1}{2} a\right)$$

hence $$Q' = \frac{1}{12} \frac{pa^2}{a' - a} \qquad \text{and} \qquad Q = \frac{1}{12} \frac{pa}{a' - a} (6\,a' - 5\,a)$$

According to these values, equation [1] of No. 135 becomes

$$\frac{EI}{\rho} = \frac{T'\,I}{v'} = \frac{1}{12} p \left(a^2 - 6\,ax + 6\,x^2\right) = \frac{1}{2} p \left(\left(\frac{1}{2} a - x\right)^2 - \frac{a^2}{12}\right)$$

It follows

1° that at point A, where $x = 0$, the moment of elasticity is $p\ a^2/12$.

2° that it then decreases and becomes zero, at the inflection point whose abscissa is given by the equation

$$\frac{1}{2} a - x = \frac{a}{2\sqrt{3}} \qquad \text{or} \qquad \frac{1}{2} a \left(1 - \frac{1}{\sqrt{3}}\right) = 0{,}211\ a$$

3° that it then becomes negative, and increases in absolute value up to point C where at $x = \frac{1}{2} a$ it becomes

$$-\frac{1}{24} pa^2 .$$

4° that beyond the midpoint C, the moment of elasticity passes back through the same preceding values.

5° that consequently the greatest curvature occurs at A.

If the uniformly loaded piece were simply supported on supports A and B, the greatest moment of elasticity (121) would be $\frac{1}{8} pa^2$, instead of $\frac{1}{12} pa^2$; it would exist at C and not at A.

139. The deflection f will be found in the case in question by substituting in formula [3] of No. 135 the above values of Q and Q', together with $P = 0$ and $x = \frac{1}{2} a$. We thus have

$$EI\, f = \frac{1}{24} p \left(\frac{a}{2}\right)^4$$

while if the piece were simply supported we would have (125)

$$EI\, f = \frac{5}{24} p \left(\frac{a}{2}\right)^4 .$$

7° Prism placed on three level supports B′, A, B, subjected to a uniformly distributed load, and to two distinct forces P′, P at the intermediate points C′, C. fig. 33.

140. Let $AB = a$, $AC = \delta$, $AB' = A'$, $AC' = \delta'$.

Call Q and Q' the unknown reactions of supports B, B'. Denote by α the angle of the tangent at A with the horizontal AB.

We will have for the curve AC conforming to the general formulas of No. 115

[1] . . . . $EI\, f''(x) = \frac{1}{2} p\,(a - x)^2 + P(\delta - x) - Q(a - x)$

[2] . . . . $EI\left(f'(x) - \tan\alpha\right) = \frac{1}{2} p \left(a^2 x - ax^2 + \frac{x^3}{3}\right) + P\left(\delta x - \frac{x^2}{2}\right) - Q\left(ax - \frac{x^2}{2}\right)$

[3] . . . . $EI\,(y - x\tan\alpha) = \frac{1}{2} p \left(\frac{a^2 x^2}{2} - \frac{ax^3}{3} + \frac{x^4}{12}\right) + P\left(\frac{\delta x^2}{2} - \frac{x^3}{6}\right) - Q\left(\frac{ax^2}{2} - \frac{x^3}{6}\right)$

and for the curve CB

[4] . . . . $EI\, f''_1(x) = \frac{1}{2} p\,(a - x)^2 + P(\delta - x) - Q(a - x)$

[5] . . . . $EI\left(f'_1(x) - \tan\alpha\right) = \frac{1}{2} p \left(a^2 x - ax^2 + \frac{x^3}{3}\right) + P\,\frac{\delta^2}{2} - Q\left(ax - \frac{x^2}{2}\right)$

[6] . . . . $EI\,(y - x\tan\alpha) = \frac{1}{2}\,p\left(\frac{a^2\,x^2}{2} - \frac{ax^3}{3} + \frac{x^4}{12}\right) + P\,\frac{\delta^3}{6} - Q\left(\frac{ax^2}{2} - \frac{x^3}{6}\right)$

This last equation must be satisfied by $x = a$ and $y = 0$, which gives the relation

[7] . . . . $EI\,\tan\alpha + \frac{1}{8}\,pa^3 + P\,\frac{\delta^2}{2}\left(1 - \frac{\delta}{3\,a}\right) - Q\,\frac{a^2}{3} = 0$

Now note that the part $AC'B'$ of the prism would give a completely similar equation, except that $tan\ \alpha$ would be replaced by $-tan\ \alpha$, a by a', δ, P and Q by δ', P' and Q': which gives

[8] . . . . $EI\,\tan\alpha + \frac{1}{8}\,pa'^3 + P'\,\frac{\delta'^2}{2}\left(1 - \frac{\delta'}{3\,a'}\right) - Q'\,\frac{a'^2}{3} = 0$.

Adding [7] and [8] we eliminate $tan\ \alpha$ and would have an equation between the two unknowns Q and Q'.

We obtain another equation by noting that by virtue of the equilibrium of the prism, we have, taking moments with respect to point A

[9] . . . . $Qa - Q'a' - p\,\frac{a^2}{2} + p\,a'^2 - P\,\delta + P'\,\delta' = 0$

The unknowns Q, Q' and $tan\ \alpha$ are thus determined by three equations of the first degree, and their values being substituted in the six preceding formulas, these will give all the circumstances of the flexure of the piece.

141. Particular case $P = 0$, $P' = 0$. Equations [7], [8] and [9] give in this case

$$Q\,a^2 + Q'\,a'^2 = \frac{3}{8}\,p\left(a^3 + a'^3\right)$$

$$Q\,a - Q'\,a' = \frac{1}{2}\,p\left(a^2 + a'^2\right)$$

hence, multiplying the 2nd by a' and adding

$$Qa\,(a + a') = \frac{1}{2}\,p\left(\left(a^2 - a'^2\right)a + \frac{3}{4}\left(a^3 + a'^3\right)\right)$$

or $$Qa = \frac{1}{2}\,p\left(a'\,(a - a') + \frac{3}{4}\left(a^2 - aa' + a'^2\right)\right)$$

or $$Q = \frac{1}{8}\,p\left(3\,a + a' - \frac{a'^2}{a}\right)$$

and by changing the accents

$$Q' = \frac{1}{8}\,p\left(3\,a' + a - \frac{a^2}{a'}\right).$$

142. One of these forces can become negative, i.e., the piece would tend to lift one of the extreme supports. For example, Q is negative if

$$a^2 - aa' > 3a^3 \quad \text{or} \quad \frac{a'}{a} > \frac{1}{2} + \sqrt{3{,}25} \quad \text{or} \quad \frac{a'}{a} > 2{,}303 .$$

Let S be the reaction of support A; we have

$$S + Q + Q' = p(a + a')$$

and by the preceding formulas

$$S = \frac{1}{2} p(a + a') \left\{ \frac{5}{4} + \frac{(a' - a)^2}{4aa'} \right\} ;$$

From this it results:

1° that the intermediate support A always carries more than half of the total load $p(a + a')$;

2° that the force S increases indefinitely as one of the distances a, a', increases, while their sum remains constant.

143. In the more particular case where a = a', we have

$$Q = Q' = \frac{3}{8} pa \quad \text{and} \quad S = \frac{10}{8} pa .$$

which is in accordance with the result of n ° 133, because in this hypothesis the part is horizontal at A.

If in the formulas [7], [8], [9] from n° 140 we make

$$a = a' , \delta = \delta' = \frac{1}{2} a , p = 0 , P = P' ,$$ we find

$$Q = \frac{5}{16} P \qquad \text{and so} \qquad S = \frac{22}{16} P$$

which agrees with the result of n° 132.

It can be seen that the method followed in n ° 140 is applicable to a prismatic part supported by any number of supports, even in the case where these supports, moreover located in the plane of the forces, would not be completely in a straight line.

8° Vertical needles supporting a water load.

144. Let l be the width of a needle, perpendicular to the plane of figure 34, and c its thickness in the direction of the water thrust. Let $AB = a$, $AC = l$, $AD = l'$, $CM = z$.

The upstream pressure on a surface element, whose height is MM' or dz and whose width is l, is expressed by $\Pi z . ldz$, where Π denotes the weight per cubic meter of water; the sum of the upstream pressures is therefore $\frac{1}{2} \Pi lh^2$. Similarly, the sum of the downstream pressures is $\frac{1}{2} \Pi l h^2$, these pressures are in equilibrium with the reactions Q, Q' of the supports B, A. We will obtain Q by taking moments about A, namely:

$$Qa = \int_0^h \Pi l z \, dz (l - z) - \int_0^{l'} \Pi l z \, dz (l' - z) = \frac{1}{6} l (h^3 l^3)$$

hence $$Q' = \frac{1}{2}\Pi\, l\,(h^2 - h'^2) - \frac{1}{6}\Pi\,\frac{l}{a}\,(h^3 - h'^3)$$

Now for any point between D and C, such as M, considering the forces acting above this point, we will have:

$$\frac{R'I}{v'} = Q\,(z + (a - h)) - \int_{x=0}^{x=x} \Pi\, a\, l\, d\, x\,(z - x)$$

or $$\frac{R'I}{v'} = Q\,(z + (a - h)) - \frac{1}{6}\Pi\, l\, z^3 \ldots\ldots\ldots\ldots [\,1\,]$$

The point of greatest curvature between C and D is obtained by setting the derivative of the second member to zero:

$$0 = Q - \frac{1}{2}\Pi\, l\, z^2,\ 0 = \frac{1}{6}\Pi\,\frac{l}{a}\,(h^3 - l'^3) - \frac{1}{2}\Pi\, l\, z^2, \text{ hence } \quad z = \sqrt{\frac{h^3 - h'^3}{3\,a}}$$

a value which is only applicable to the question insofar as it is found to be smaller than CD or h - h'.

This condition being verified, we will have the expression for the greatest moment of elasticity by substituting the value of z above into the preceding formula [1].

In the opposite case, the point of greatest flexure between A and D must be sought.

Let y be the distance Am' from any point of the curve AD to point A. The equilibrium of this part under the action of the force Q' calculated above, and of the upstream and downstream pressures, will give:

$$\frac{R'I}{v'} = Q'y - \Pi\,(h - h')\, l\, y\,.\,y^2,$$

and the maximum flexure will correspond to the equation

$$Q' - \Pi\,(h - h')\, l\, y = 0$$

hence $$y = \frac{h + h'}{2} - \frac{h^3 - h'^3}{6\,a\,(h - h')} \quad \text{and} \quad \frac{R'I}{v'} = \frac{Q'y}{2};$$

finally, eliminating Q' and y, we obtain the expression for the maximum moment of elasticity between A and B, namely:

$$\frac{R'I}{v'} = \frac{\Pi\, l\,(h - h')}{2}\left(\frac{h + h'}{2} - \frac{h^3 - h'^3}{6\,a\,(h - h')}\right)^2$$

which will only be applicable insofar as the preceding value of y is smaller than h'.

145. One can avoid the integrations, moreover very easy, of the previous number by means of a known geometrical construction where the pressure exerted by the water on a plane is represented by a prism whose center of gravity corresponds to the center of pressure.

146. It is now supposed that the vertical needles supporting a water load bear against three horizontal and equidistant crosspieces A, B, C (fig. 35). The upstream water level is flush with

the upper support C and there is no water downstream. The pressures Q, S , Q' exerted by the supports C, B, A are required.

Let $AB = BC = a$ and $cm = x$. At any point m of the curve CB, taking the dimension perpendicular to the figure as unity, we have:

$$EI\, f''(x) = \frac{1}{6}\,\Pi\, x^3 - Q\, x$$

Integrating, we obtain:

$$EI\, f'(x) = C + \frac{1}{24}\,\Pi\, x^4 - \frac{1}{2}\,Q\, x^2$$

for $x = a$, let $f'(x) = \tan\alpha$, the slope of the curve at B.

Thus: $$EI\, \tan\alpha = C + \frac{1}{24}\,\Pi\, a^4 - \frac{1}{2}\,Q\, a^2 .$$

hence: $$EI\left(f'(x) - \tan\alpha\right) = -\frac{1}{24}\,\Pi\left(a^4 - x^4\right) + \frac{1}{2}\,Q\left(a^2 - x^2\right).$$

Integrating again, given that x and y are simultaneously zero, we have:

$$EI\,(y - x\tan\alpha) = -\frac{\Pi}{30}\,a^4 + \frac{1}{3}\,Q\,a^2 \;\ldots\ldots\;[\,1\,]$$

Now at any point m' of the curve AB, taking the origin of y' and x' at A and consequently $Am' = x'$, we have:

$$EI\, f''_1(x') = \Pi\, a\, x'^2 - \frac{1}{6}\,\Pi\, x'^3 - Q'\, x'$$

which is seen by representing the pressure on Am' by the rectangle $ADEm'$ minus the triangle DEF. Integration gives:

$$EI\, f'_1(x') = \Pi\left(\frac{1}{3}\,a\,x'^3 - \frac{1}{24}\,x'^4\right) - \frac{1}{2}\,Q'\,x'^2 + C .$$

for $x' = a$ we have $f'_1(a) = -\tan\alpha$; therefore, eliminating C

$$EI\left(f'_1(x') + tang\ \alpha\right) = -\Pi\left(\frac{a}{3}\left(a^3 - x'^3\right) - \frac{1}{24}\left(a^4 - x'^4\right)\right) + \frac{1}{2}\,Q'\left(a^2 - x'^2\right)$$

The integral of this equation, given that $x' = 0$ and $y = 0$ simultaneously, is:

$$EI\,(y' + x'\tan\alpha) = -\Pi\left(\frac{a}{3}\left(a^3\,x' - \frac{x'^4}{4}\right) - \frac{1}{24}\left(a^4\,x' - \frac{x'^5}{5}\right)\right) + \frac{1}{2}\,Q'\left(a^2\,x - \frac{x'^3}{3}\right).$$

y must still become zero at $x' = a$, which gives:

$$EI\,\tan\alpha = -\Pi\,\frac{13}{60}\,a^4 + \frac{1}{3}\,Q'\,a^2 \;\ldots\ldots\ldots\ldots\;[\,2\,]$$

Eliminating $tan = \alpha$ from the two equations [1] and [2] we obtain

$$Q + Q' = \frac{3}{4} \Pi a^2$$

Moreover, we have the total pressure on the supports:

$$Q + Q' + S = 2 \Pi a^2$$

And taking moments of all forces about B, we find:

$$(Q - Q')\, a = 2 \Pi a^{2.} \frac{a}{3} \qquad \text{or} \qquad Q' - Q = \frac{2}{3} \Pi a^2$$

We thus have three very simple equations between the three forces Q, Q' and S . If we denote by P the total water pressure $2 \Pi a^2$, we have:

$$Q' = \frac{17}{48} P, \qquad Q = \frac{1}{48} P, \qquad S = \frac{30}{48} P.$$

The forces supported by the pieces *A, B, C* being thus known, we can determine the dimensions suitable for them (123). One could propose and solve the question of knowing at what distance support *B* should be located behind the vertical AC for the loads S and Q to be equal.

§: 5. Experimental data and applications.

147. We extract the tables below from the interesting and very useful work entitled Études théoriques and expérimentales sur l'établissement des charpentes à grande portée by Mr. Ardant, officer of the engineers, referring, as he himself does, to the well-known works of Messrs. Navier, Poncelet and Morin.

The last four columns of the second table relate to prisms of great length compressed along this length. Known theories on this subject relate only to very thin prisms and do not apply to pieces used in constructions.

I Table of the resistance of bodies to extension

Designation of substances	Coefficient of elasticity E divided by 1,000,000	Tension per square millimeter producing rupture	R_1 divided by 1,000,000 or limit of permanent forces per square millimeter
Oak wood	1,200	6	0.50 to 0.70
Fir wood	1,300	8	0.60 to 0.80
Arches in planks of timber	500	3	0.25 to 0.30
Iron wire in cables	18,000	30	6 to 10 according
Wrought iron above 0.06 m side	18,000	40	4 to 8 to
Wrought iron below 0.06 m side	20,000	60	6 to 10 quality
Grey cast iron not exposed to shocks	12,000	12.5	7.5

II Table of resistance to crushing

Designation of substances	Weight per cubic meter	Weight producing crushing of a 0.01 m cube	Limit of permanent loads per million. q the ratio of length to small side of the base being				
			below 12	12	24	48	60
Strong Oak	980 kg	300 kg	0.3 kg	0.25	0.15	0.05	0.025
Weak Oak	900	190	0.19	0.084	0.056	"	"
Yellow or Red Pine	671	375	0.375	0.31	0.187	0.075	"
White Pine	550	97	0.097	0.082	0.049	"	"
Wrought Iron	7,783	4,900	10.	8.35	5.	1.67	0.84
Cast Iron	7,202	10,000	20.	16.70	10.	3.33	1.67

148. An oak piece, built-in horizontally at one end, is loaded at the other end with a weight P; its projecting length is a, its cross-section is l wide and c high. Determine the practical maximum of P considering the own weight of the piece, which is Π per cubic meter. The formulas are (117 and 102):

$$\frac{R'I}{v}' = \left(\frac{1}{2}pa + P\right)a\,, \qquad I = \frac{1}{12}lc^3\,,$$

in which we must set: $\mathrm{v}' = \frac{1}{2}C\,, \qquad pa = \Pi\, a\, l\, c\,,$

Thus:

$P + \frac{1}{2}\Pi\, abc = \frac{1}{6}\,\frac{b\,c^2}{a}\,R'$, where we would take $R' = 0{,}6 \times 10^6$ and approximately $\Pi = 900$

If the amount by which the free end lowers is required, we have (119):

$$h = \frac{1}{E\,I}\left(\frac{p\,a}{8} + \frac{P}{3}\right)a^3 = \frac{12\,a^3}{b\,c^3\,E}\left(\frac{\Pi\,abc}{8} + \frac{P}{3}\right).$$

and we must take $E = 1200 \times 10^6$.

149. A sluice gate being closed by horizontal beams, the total head is H, the distance between supports is a; determine the thickness c of the beams. Those below the downstream water level support a load due to height H. The formula, setting $P = 0$ and $R_1 = R$, is therefore that of No. 121:

$$\frac{R_1 I}{V'} = \frac{1}{8}\,p\,a^2\,;$$

calling h the vertical dimension of the beam, and Π the weight per cubic meter of water, we have

$$p\,a = \Pi\, H\, ab\,; \qquad I = \frac{1}{12}\,b\,c^3 \qquad \text{and} \quad v' = \frac{1}{2}\,C\,.$$

The above formula therefore becomes:

$$\frac{1}{6} R_1 b c^2 = \frac{1}{8} \Pi H a^2 b \qquad \text{hence} \qquad C^2 = \frac{3}{4} \frac{\Pi H a^2}{R_1}$$

If the beams are of oak, one can reduce their cross-section as much as possible, taking $R_1 = 0.8 \times 10^6$; if they are of fir, one will take $R_1 = 0.9 \times 10^6$.

§: 6. Solids of equal resistance.

150. When a solid offers a great length, relative to its transverse dimensions, but these vary instead of remaining constant as we supposed in §: 2 and following, if this variation occurs in barely perceptible degrees, one can conceive the length divided into several parts for each of which the piece differs little from a prism. Consequently, if one considers a section aa' (fig. 16), beyond which the piece is subjected to forces sensibly parallel to this section, one can admit that the equation $\sum M\{F\}_0 = \sum M\{A\}_0$ still holds, and that the moment of elasticity $\sum M\{A\}_0$ can still be expressed by

$$\frac{R' I}{V'} \qquad \text{or by} \qquad E I \int''(x)$$

as in Nos. 99 and 101, calling I the moment of inertia of the section aa' with respect to an axis o passing through the center of gravity and perpendicular to the direction of the forces F, and denoting by $\int(x)$ the extent of the displacement of any point of the straight or slightly curved line passing through the centers of gravity of the cross-sections.

151. Consider the case where the piece, of variable section and slight curvature, is approximately horizontal, maintained in this situation at A near one of its ends by supports, and acted upon at the other end B by a single vertical force P, so that its own weight is neglected here. If we set (fig. 36), $AB = l$ and $AM = x$, we have at M:

$$\frac{R' I}{V'} = P(a - x)$$

an equation which differs from that which one would have for a prism, in that I and v' are variable.

Suppose, for example, that the cross-sections of the piece are all rectangular, so that the width b perpendicular to the plane of the figure is constant while the height aa' is variable and represented by z. We then have (102) . . . . . $I = \frac{1}{12} b z^3$ and $V' = \frac{1}{2} z$.

The preceding equation therefore becomes

$$\frac{R' b z^2}{6} = P(a - x),$$

and one can vary z so that R', the maximum resistance of the section corresponding to the abscissa x, is constant throughout the extent AB. The piece is then called a solid of equal

resistance relative to any force acting like P at the end B. For this, z^2 must be proportional to $l - x$, so that if the line $A\ M\ B$ of the centers of gravity is straight in the natural state, the curve a $B\ a'$, whose variable ordinate M a is $\frac{1}{2}z$, is a parabola of the second degree whose vertex is at B.

One can, without changing the property of the solid, dispose of one of the curves a B, $a'\ B$ provided it differs little from a straight line.

152. If the cross-section were circular with a variable radius z, we would have (110)

$$I = \tfrac{1}{4}\,\Pi\, R'\, z^4 \qquad \text{and} \qquad V' = z$$

$$\text{hence} \qquad \frac{1}{4}\,\Pi\, R'\, z^3 = P\,(a - x)$$

equation of the meridian curve of the piece, supposed to be a solid of revolution.

In both preceding hypotheses, if one wished to determine the smallest transverse dimensions corresponding to a given force P, it would be necessary to assign to R' its maximum R_1, depending on the substance of the body.

153. If the piece, held horizontally at A, supports from A to B a uniformly distributed load pa, we have at any point M

$$\frac{R'\,I}{V'} = \frac{1}{2}\,p\,(a - x)^2$$

$$\text{so, as in No. 151,} \qquad I = \frac{1}{12}\,h\,z^3 \quad \text{and} \quad V' = \frac{1}{2}\,z$$

$$\text{hence} \qquad \frac{1}{6}\,R'\,h\,z^2 = \frac{1}{2}\,p\,(a - x)^2$$

The solid will be of equal resistance relative to this distribution of the load if x is proportional to $a - x$; consequently if the mean line AMB is straight, the lines aB, $a'B$, will also be straight lines converging at B.

§: 7. Prisms subjected to forces of any directions.

154. Let a piece whose portion $a\ a'\ B$ (fig. 37), naturally straight or curved, be slightly deformed by the action of one or more forces. Let us again call A the various reactions normal to the plane aa', and suppose that the resultant of the other arbitrary forces applied to the piece is decomposed into two forces, one N normal to aa', the other parallel to this same section. Let a be the distance of P from this plane and let y be the distance of N from the center of gravity G of aa'.

The various forces A and the two forces P and Q satisfy the two equations

$$\sum A = N \qquad \text{and} \qquad \sum M\,\{A\}_6 = P\,a + N\,y$$

for one can take for the axis of moments any line in the plane aa', and consequently the axis G, although the invariable fiber no longer passes in this case through the center of gravity, as will be seen.

Given that the sliding of section bb', very close to aa', can be considered zero, we will still admit that each force A for an element $d\omega$ is proportional to the distance of this element from a line projected at O where the normal reaction is zero. Thus, by designating by V the distance OG and by γ, as before, the distance from the element $d\omega$ to the line G, we will have :

$$A = \varphi\,(\gamma + V)\,d\,\omega \qquad \text{hence} \qquad \sum A = \varphi\, V\, \Omega \qquad \text{because} \quad \sum \gamma\, d\,\omega = 0$$

we therefore have $$\varphi\, V\, \Omega = N \;\ldots\ldots\ldots\ldots [\,1\,]$$

and V is zero only when N is zero.

Now the moment of one of the forces A with respect to G is $A\,\gamma$ or $\varphi\,(\gamma + V)\,\gamma\,d\,\omega$; therefore

$$\sum M\,\{A\}_G = \varphi \sum \gamma^2\, d\,\omega \quad (\text{ because } \varphi\, V \sum \gamma\, d\,\omega = 0\,)$$

we thus have for the moment of elasticity with respect to the axis G the same expression as when the invariable fiber passes through this axis, namely

$$\sum M\,\{A\}_G = \varphi\, I$$

thus the moment equation becomes $\varphi\, I = P\,a + N\,y \;\ldots\ldots\ldots\ldots [\,2\,]$

Finally, the quantity φ, calling R' the resistance per unit area at the point which is at distance γ' from the center of gravity, satisfies the relation

$$R'\,d\,\omega = \varphi\,(\gamma' + V)\,d\,\omega\,, \quad \text{hence} \qquad \varphi = \frac{R'}{\gamma' + V} \;\ldots\ldots [\,3\,];$$

so that if the quantities P, N, a, y, Ω, I and γ' are given, we have three equations for the three unknowns V, φ, R'. From them we conclude

$$\varphi = \frac{P\,a + N\,y}{I} \;\ldots [\,4\,] \qquad V = \frac{N\,I}{\left(P\,a + N\,y\right)\Omega} \;\ldots [\,5\,] \qquad R' = \frac{\left(P\,a + N\,y\right)\gamma'}{I} + \frac{N}{\Omega} \;\ldots [\,6\,]$$

In [4] we can replace φ as previously by $\frac{E}{\varsigma}$, calling ρ the distance at which two infinitely close sections that would be parallel in the natural state meet. Thus the change of curvature or the flexure of the piece at section aa' depends only on the moment of the resultant of the forces P and N with respect to G, whatever the direction of this resultant.

But equation [6] shows that it is not the same for the resistance R', which is composed of two parts, one due to the moment $P\,a + N\,y$ and the other due to the normal component. This is what Navier stated (page 248 of his lessons) without demonstrating it.

155. Particular case: vertical prism loaded with a weight acting at a certain distance from the axis of this piece. Let (fig. 38)

a be the height OA of a prism vertically built-in at O,

l the distance BC between the end B and the point C where the weight N is applied,

f the deflection AB produced by this weight,

y, x the coordinates of a point M of the bent axis, with respect to the axes O_y, O_x, one horizontal, the other vertical.

If we neglect the weight of the piece itself, the moment of elasticity at M is equal to the moment of N with respect to the center of gravity M (154), and we have

$$E\,I\,f''(x) = N\left(l + f - y\right)$$

To integrate this equation, let us write it in the form

$$\frac{d\,p}{d\,x} = K\,(b - y)\,;$$

setting . . . . . $p = \frac{d\,y}{d\,x}$

hence, multiplying . . . . . $p\,d\,p = K\,(b - y)\,d\,y$

and integrating . . . . . $p^2 = K\left(2\,b\,y - y^2\right)$

without a constant since p and y are zero simultaneously.

The last equation becomes

$$\sqrt{K}\,d\,x = \frac{d\,y}{\sqrt{2\,b\,y - y^2}} \qquad \text{or} \qquad \sqrt{K}\,d\,x = \frac{d\,y}{\sqrt{b^2 - (b - y)^2}}$$

hence, integrating $x\,\sqrt{K} = arc\left(\cos 1 - \frac{y}{b}\right)$

without another constant since y and x are simultaneously zero. This equation is equivalent to

$$1 - \frac{y}{b} = \cos x\,\sqrt{K} \qquad \text{or} \qquad y = (l + f)\left(1 - \cos x\,\sqrt{\frac{N}{E\,I}}\right)$$

Now for $x = a$, y becomes f; from which we conclude

$$f = l\left(\frac{1}{\cos a\,\sqrt{\frac{N}{E\,I}}}\right) - 1 \qquad \ldots\ldots\ldots [\,1\,]$$

formula in which it is easy to see that a $a\,\sqrt{\frac{N}{E\,I}}$ is an abstract number.

Knowing f, we will apply formula [6] of the previous number by setting

$$R' = N\,(l + f)\,\frac{V'}{I} + \frac{N}{\Omega} \qquad \ldots\ldots\ldots [\,2\,]$$

If the only unknowns were f and R', they would be determined immediately; but if it were a question of determining the cross-section of the piece so that a, P, and l being given, the

resistance R' does not exceed an assigned limit, the unknown would be more difficult to isolate because it would enter into the angle $a\sqrt{\frac{N}{EI}}$. In this case, we will note that this angle is small because the ratio $\frac{f}{l}$ equal to $\frac{1}{\cos\sqrt{\frac{P}{E}}} - 1$ is itself small and consequently the cosine differs little from unity. Now, a small arc can be confused with its chord, we have approximately

$$(arc)^2 = 2\,(1 - cosinus)$$

hence $$\cos = 1 - \frac{1}{2}\,(arc)^2$$

formula [1] can therefore be replaced by

$$f = \left(\frac{1}{1 - \frac{1}{2}\,\frac{a^2 N}{E I}} - 1 \right) l \quad \ldots\ldots\ldots [\,3\,]$$

and consequently equation [2] gives

$$R' = \frac{N \gamma' l}{I}\,\frac{1}{1 - \frac{1}{2}\,\frac{a^2 N}{E I}} + \frac{N}{\Omega} \quad \ldots\ldots [\,4\,]$$

If the section is rectangular we have $\Omega = bc$, $I = \frac{1}{12}\,bc^3$ and $V' = \frac{1}{2}\,c$; if it is circular $\Omega = \Pi\, r^2$, $I = \frac{\Pi\, r^4}{4}$ and $V' = r$.

Thus equation [2] becomes

for a rectangular prism $$bc^2 = \frac{N}{R'}\left(c + \frac{6\,l}{1 - \frac{6\,N\,a^2}{E\,b\,c^3}} \right)$$

for a cylinder $$r^2 = \frac{N}{\Pi\,R^2}\left(r + \frac{4\,l}{1 - \frac{2\,N\,a^2}{E\,\Pi\,r^4}} \right)$$

Example taken from the cited work of Mr. Ardant (page 105). For cylinder, $a = 15\ m$ $l = 1$; $N = 5000^{Kil}$; $R' = 6 \times 10^5$; $E = 12 \times 10^8$; we find the diameter $2r = 0.49\ m$ and the deflection $f = 0.166\ m$. It would be more accurate to take into account the weight p of the piece by writing $\frac{N + pa}{\Omega}$ for the last term of formula [4] instead of $\frac{N}{\Omega}$, which amounts to neglecting the influence of $p\,a$ on the flexure of the piece.

156. One can obtain in a simpler manner a sufficiently approximate solution of the preceding question by neglecting the difference $f - y$ in the first equation and setting

$$E\,I\,f''(x) = N\,l \qquad \text{hence} \quad E\,I\,f'(x) = N\,l\,x \quad \text{then } E\,I\,y = \frac{1}{2}\,N\,l\,x^2$$

and consequently the deflection $$f = \frac{1}{2}\,\frac{N\,l\,a^2}{E\,I}$$

and by substituting P and f for N and y in equation [6] of No. 154

$$R'\Omega = N(l+f)\frac{V'\Omega}{I} + N \qquad \text{or} \qquad R' = \frac{N v' l}{I}\left(1 + \frac{N l a^2}{2 E I}\right) + \frac{N}{\Omega}.$$

One would obtain the same results from equations [3] and [4] above by replacing the fraction $\frac{1}{1-\frac{1}{2}\frac{a^2 N}{E I}}$ by $1+\frac{1}{2}\frac{a^2 N}{E I}$ which differs very little from it, given that the deflection is always very small compared to l.

157. If we modified the question of No. 155 by adding a force P parallel to BC, the equation relating to flexure would be

$$E I f''(x) = P(a-x) + N(l+f-y) \quad \ldots\ldots [\,1\,]$$

an equation which is integrated by noting on the one hand that if we replace $a - x$ by $-x'$ and $l + f - y$ by y', this equation becomes

$$E I \frac{d^2 y'}{d x^2} = -P x' - N y'$$

and on the other hand that a function of the form $y = A x + B \sin C x + D \cos C x$

gives $$\frac{d y}{d x} = A + C B \cos C x \; C D \sin C x$$

and $$\frac{d^2 y}{d x^2} - C^2 B \sin C x - C^2 D \cos C x = C^2 A x - C^2 y$$

which becomes identical to the above equation in y' and x' when we set

$$C^2 A = \frac{P}{E I} \qquad \text{and} \qquad C^2 = \frac{+N}{E I} \qquad \text{hence} \qquad A = -\frac{P}{N}$$

the general integral is therefore

$$y' = -\frac{P}{N} x' + B \sin x' \sqrt{\frac{N}{E I}} + D \cos x' \sqrt{\frac{N}{E I}}$$

and the two constants B, D are determined by the condition that for $x = 0$, we have $y = 0$ and $\frac{d y}{d x} = 0$.

Instead of pursuing this calculation further, on which one can consult Navier's lessons, we will note that equation [1] would be simplified by neglecting, as in No. 156, the difference $f - y$; we would thus have approximately

$$E I f''(x) = P(a-x) + N l$$

whose integral $$E I y = (P a + N l)\frac{x^2}{2} - \frac{P x^3}{6}$$

would give for $x = a$ the deflection $f = \frac{1}{E I}\left(\frac{N l a^2}{2} + \frac{P a^3}{3}\right)$

a value which it would be necessary to substitute in the following formula, conforming to equation [6] of No. 154

$$R' = \left(P a + N (l + f)\right) \frac{V'}{I} + \frac{N}{\Omega}$$

to determine either the extreme resistance R', or the transverse dimensions of the piece if this resistance were given.

§: 8. *Flexure and resistance of curved pieces.*

158. A naturally curved piece being subjected to known external forces, two questions can be proposed: 1° Determine the change in shape of the piece. 2° Determine the extreme resistance R' in a designated section. If the intensities of the forces and their positions relative to this section were known, formula [6] of No. 154 would immediately provide the solution to the second question. We will now deal with the first question in the case where the forces are situated in the plane of the original curve of the piece, this plane supposed vertical (fig. 39). Let aa' be any cross-section and bb' another infinitely close section, in the natural state. Suppose that in the state of flexure this second section takes, relative to the first, the position $b_1 b'_1$; it follows that the elementary fiber which naturally had the length mm' has undergone the variation m'm$_1$ and consequently the force A corresponding to this fiber is $A = E\, d\, \omega \frac{m' m_1}{m\, m'}$.

Call α and $\alpha + d\,\alpha$ the angles that the sections aa', bb' make with the vertical in the state of flexure; denote by $d\,\delta$ the length oo' supposed invariable: we will have, setting $om = z$,

in the natural state . . . . . $m\, m' = d\,\delta - z\, d\,\alpha$

in the state of flexure . . . . . $m\, m_1 = d\,\delta - z\, d\,\alpha'$

therefore $m' m_1 = z\,(d\,\alpha' - d\,\alpha)$, therefore $A = E\, d\, \omega \frac{z\,(d\,\alpha' - d\,\alpha)}{d\,\delta - z\, d\,\alpha}$

or, neglecting $z\, d\,\alpha$ compared to $d\,\delta$, given the slight curvature of the piece

$$A = E\, d\, \omega \frac{z\,(d\,\alpha' - d\,\alpha)}{d\,\delta}$$

Now, calling v as before the distance from the element $d\,\omega$ to the projection G of the center of gravity and designating by V the distance OG, we have $z = v \pm V$ and

$$\sum M\,\{A\}_G = \sum AV = \frac{E\,(d\,\alpha' - d\,\alpha)}{d\,\delta} \sum v^2\, d\,\omega$$

since $V \sum v\, d\,\omega = 0$. Denoting as usual by I the moment of inertia of the section aa' with respect to the axis G, this formula becomes

$$\sum M\,\{A\}_G = E\, I \frac{d\,\alpha' - d\,\alpha}{d\,\delta}.$$

That established, designating by F any one of the forces acting on the piece from the section aa' up to its end, we will have, taking the moments of all forces F with respect to the axis G

$$\frac{d\,\alpha' - d\,\alpha}{d\,\delta} = \frac{1}{E\,I}\sum M\{F\}_G .$$

The second member of this equation can be considered as an implicit or explicit function of δ, in which the forces enter as constants or as functions of δ. The moments of these forces with respect to point G at the end of δ should be taken in the state of flexure of the piece; but since this flexure is very slight, one can, without appreciable error, calculate the moments of the forces according to their positions in the natural state of the piece.

Multiplying both members of the equation by $d\,\delta$ and integrating, we will have

$$(\alpha' - \alpha) = \frac{\int_0^{\delta} d\,\delta \sum M\{F\}_G}{E\,I}$$

the integral being taken up to the point G, end of any arc δ, from an origin where the direction of the curve is supposed the same in the state of flexure as in the natural state. We thus have the variation α' - α of the angle of the normal at the end of δ, with the vertical, due to flexure.

To pass from this first result to the determination of the variation of the position of point G, we employ the following device, due to Navier, after noting that the above integral can be considered either as a function of δ, or as a function of x, which we will denote for brevity by *F(x),* calling x and y the horizontal and vertical coordinates of point G, in the natural state; thus $\alpha' - \alpha = F(x)$;

the angle $\alpha' - \alpha$ being very small, we can write approximately

$$\sin(\alpha' - \alpha) = \alpha' - \alpha \qquad \text{and} \qquad \cos(\alpha' - \alpha) = 1$$

Now, considering α' as equal to $(\alpha' - \alpha) + \alpha$ we have

$$\cos\alpha' = \cos(\alpha' - \alpha)\cos\alpha - \sin(\alpha' - \alpha)\sin\alpha$$
$$\sin\alpha' = \sin(\alpha' - \alpha)\cos\alpha + \cos(\alpha' - \alpha)\sin\alpha$$

hence, substituting $\cos(\alpha' - \alpha) = 1$, $\sin(\alpha' - \alpha) = F(x)$, then noting that if we denote by x', y' the coordinates in the state of flexure, of the same point G whose coordinates in the natural state are x, y, we have

$$\cos\alpha' = \frac{d\,x'}{d\,\delta} \text{ and } \sin\alpha' = \frac{d\,y'}{d} \text{ just as } \cos\alpha = \frac{d\,x}{d\,\delta} \text{ and } \sin\alpha = \frac{d\,y}{d\,\delta}$$

we conclude

$$d\,x' - d\,x = -d\,y\,F(x) \qquad \text{and} \qquad d\,y' - d\,y = d\,x\,F(x)$$

and integrating

$$x' - x = -\int_0^x d\,y\,F(x) \qquad \text{and} \qquad y' - y = \int_0^x d\,x\,F(x)\ ;$$

thus the displacements of point G, one horizontal x' - x, the other vertical y' - y, are determined by two quadratures, for the original curve being given, y is a function of x, say $y = f(x)$ and we have $d\,y = f'(x).\,d\,x$. This derivative *f'(x)* expresses the variable *tan* α.

159. Remark and approximate method. The quantity $\sum M\{F\}_G$ may not be expressible by a general function of δ. But one can always represent graphically the integral $\frac{\int_0^\delta d\,\delta \sum M\{F\}_G}{EI}$ by the area of a curve, by taking as abscissas the lengths δ from the origin of the curved piece to the various points G, and as ordinates the quotients obtained by dividing the moments of the forces, acting beyond the successive points G, with respect to these same points, by E I. It will be better, for the rest of the calculation, to take as abscissas the successive values of x; for this purpose, we substitute $d\,\delta = \frac{d\,x}{\cos\alpha}$ and consequently plot as ordinates the values of

$$\frac{\sum M\{F\}_G}{EI\cos\alpha}$$

The areas of this curve, counted from the origin, will be the values of *F(x)*. We will calculate these values numerically, then represent them graphically by lines plotted as ordinates at the ends of the abscissas x. This will result in a new curve whose area for any abscissa x will be the corresponding value of

$$\int_0^x d\,x\,F(x) \quad \text{or} \quad (y' - y).$$

Finally, the same values of *F(x)* being multiplied by the corresponding values of $\frac{d\,y}{d\,x}$ or *tang* α, and the products obtained being again plotted as ordinates at the ends of the abscissas x, this will result in a 3rd curve whose area, for an abscissa x, will be the corresponding value of

$$\int_0^x x\,.\,tang\,\alpha\,F(x) \quad \text{or} \quad \int_0^x d\,y\,F(x) \quad \text{or} \quad (x' - x).$$

If the cross-section of the piece were constant, it would be appropriate to place the divisor E I outside the $\int$ sign and modify the preceding indications accordingly. In all cases, we see that a calculation, perhaps long, but without difficulty, will provide the means to determine the vertical and horizontal displacements of the various points of a given curved piece subjected to given forces.

160. The practice of the Engineer's art presents cases where some of the forces which act on curved pieces must be determined by calculation; for example, an arch which in its natural state has a known chord length is placed on two level supports which make this chord invariable while the arch is subjected to given vertical loads: to ensure that this piece will suitably resist the forces, it is necessary first to calculate the reactions of the supports; their vertical components are immediately known, but their horizontal components must be deduced from the condition of the invariability of the chord; and one can always achieve this by a trial and error method easy to find from the preceding.

161. *Particular cases*. It may happen that a curved piece subjected to external forces nevertheless retains the same shape it would have in the natural state, except for the very slight effect of the lengthening or shortening of the mean fiber. For this, it is necessary and

sufficient that if we divide the piece into two parts by any section, the algebraic sum of the moments of the forces applied to one of these parts, with respect to the center G of gravity of this section, be zero. The resultant of these same forces, passing through point G, is the pressure or tension in the section considered.

This consideration holds when the arch, having the shape of a catenary, supports parallel loads uniformly distributed over its length, and the extreme points receive forces tangent to the curve.

It is also satisfied when the arch, having the shape of a parabola whose principal axis is vertical, supports vertical loads uniformly distributed over the horizontal projection of this arch, and the extreme forces are tangent to the curve. In this case, the formulas relating to suspension bridges (Nos. 51 and seq.) are applicable, and the equation of the curve being $y = ax^2$, if we denote by T_x the horizontal thrust and by p the load per unit horizontal length, we have $T_x = \frac{p}{2\alpha}$.(*)

Consequently, when it concerns an arch whose horizontal chord will be $2c$ and whose sag will be h, we will have $h = \alpha c^2$, hence

$$T_x = \frac{p\, c^2}{2\, h}$$

the horizontal component of the resistance of each support, while its vertical component will be

$$T_y = p\, c$$

and consequently the total resistance $\sqrt{T_x^2 + T_y^2} = \frac{p\, c^2}{2\, h}\sqrt{1 + \frac{4\, h^2}{c^2}}$.

These results are applicable to circular arc curves whose sag is small relative to the chord. Mr. Ardant, making use of Navier's analysis, finds for this case (page 120 of the cited work) for the horizontal thrust a formula which reduces to this one

$$T_x = \frac{p\left(c^2 + h^2\right)}{2\, h}$$

and which consequently gives a value slightly larger than the above expression, relating to the parabolic curve.

One will find on the subject of this paragraph learned and useful developments in the lessons of Mr. Navier and in the studies of Mr. Ardant.

(*) We find this formula very easily by considering the equilibrium of a portion of arc from the vertex where the tangent is horizontal; the vertical forces give $T_y = px$ and the equation of moments with respect to the origin $T_y \,.\, x = T_x\, y + \frac{1}{2}\, p\, x^2$ from which, by eliminating T_y, we conclude $y\, T_x = \frac{1}{2}\, p\, x^2$ or $\alpha\, T_x = \frac{1}{2}\, p$.

§: 9. *Torsion of prisms*

162. A prism being fixed invarably near one of its ends, and acted upon at the other end by various forces equivalent to a couple situated in a cross-section, the equilibrium which is established requires that every section aa' exert on its neighbor bb' forces reducible to a couple parallel to the first and of the same moment. These forces result from the displacement of one of these sections relative to the other, a displacement which consists of a rotational movement about an interior axis of the prism.

Let Θ be the angle of torsion of the piece per unit length;

$d\omega$ the common area of two superficial elements which in the natural state would be opposite each other in the planes aa', bb';

r their distance from the axis, which does not change in the state of torsion.

We satisfy the observed phenomena by admitting that the element $d\omega$ of plane bb' is acted upon to return to its natural position by a force perpendicular to the radius r and proportional

1° to the area $d\omega$,

2° to the angle of torsion Θ per unit length, since, all other things being equal, the relative displacement of the two considered elements is proportional to this angle;

3° to the distance r, for the same reason.

Thus we can express this force by . . . . . $G\,\Theta\,r\,d\omega$

and its moment, with respect to the axis of the prism, by . . . . $G\,\Theta\,r^2\,d\omega$, calling G a coefficient to be determined by experiment according to the substance of the prism, and whose numerical value depends on the units to which the angle Θ and the linear dimensions are referred.

Therefore, denoting by PR the moment of the couple acting externally on the prism, we have

$$PR = G\,\Theta \sum r^2\,d\omega \quad \ldots\ldots\ldots [1]$$

163. When the section of the prism is a circle or a regular polygon, it is evident that the axis of torsion passes through the center of figure. It can be shown that it always passes through the center of gravity. Indeed, the forces $G\,\Theta\,r\,,\ d\omega$ must be equivalent to a couple, and consequently the algebraic sum of their projections onto any line is zero.

Let 0 be the center; denoting for brevity the force $G\,\Theta\,r\,d\omega$ by F, we have

$$\sum F_x = 0 \quad \text{and} \quad F_x = F\,.\,\frac{y}{r}$$

$$\text{therefore} \quad \sum F\,\frac{y}{r} = 0 \quad \text{therefore} \quad \sum G\,\Theta\,y\,d\omega = 0 \quad \text{or} \quad G\,\Theta\,y\,d\omega = 0$$

hence the center of gravity is on the axis Ox. It is also on Oy, and consequently at O.

164. If we denote by $T\,d\,\omega$ the force or torsional resistance that the element $d\,\omega$ of plane aa' exerts on the solid, we have

$$T\,d\,\omega = G\,\Theta\,r\,d\,\omega \qquad \text{hence} \qquad \frac{T}{r} = G\,\Theta\,,$$

a constant ratio for a given state of torsion. Let r' be the greatest distance r, T takes a corresponding value T' which is the greatest tension in the section aa'.

Substituting $\frac{T'}{r'}$ for $G\,\Theta$ in formula [1], we have

$$P\,R = \frac{T'}{r'}\sum r^2\,d\,\omega \quad \ldots\ldots\ldots [\,2\,]$$

165. When the section is a circle of radius r', we have

$$\sum r^2\,d\,\omega = \int_0^{r'} r^2\,.\,2\,\Pi\,r\,d\,r = \frac{1}{2}\,\Pi\,r'^4 \quad \text{hence} \quad P\,R = G\,\Theta\,\frac{\Pi\,r'^4}{2} = T'.\,\frac{1}{2}\,\Pi\,r'^3$$

If it is a zone between two concentric circles, we have

$$\sum r^2\,d\,\omega = \frac{1}{2}\,\Pi\left(r'^4 - r''^4\right).$$

If it is a rectangle whose sides are b and c, we have, by passing through the center two axes Ox, Oy, parallel to the sides

$$\sum r^2\,d\,\omega = \sum\left(x^2+y^2\right)d\,\omega = \int_{-\frac{b}{2}}^{\frac{b}{2}} x^2\,.\,c\,d\,x + \int_{-\frac{c}{2}}^{\frac{c}{2}} y^2\,b\,d\,y = \frac{1}{12}\,b\,c\left(b^2+c^2\right)$$

This formula, in the case of a square, reduces to

$$\sum r^2\,d\,\omega = \frac{1}{6}\,b^4 \quad \text{hence} \quad P\,R = G\,\Theta\,\frac{b^4}{6} = \frac{T'\,b^3}{3\sqrt{2}} = \frac{T'\,b^3}{4{,}24}$$

166. *Experimental data*. It follows from Duleau's experiments on the torsion of wrought iron that we have approximately, taking the meter as unit

For round iron	$G = 66\times10^8$	when the angle Θ is expressed by the ratio of the arc which measures it to the radius.
For square iron	$G = 55\times10^8$	
For round iron	$G = 115\times10^6$	when the angle Θ is expressed in degrees.
For square iron	$G = 96\times10^6$	

Example: round iron; diameter 0.02; length 3.17; total torsion angle 6"

$$P\,R = G\,\Theta\,.\,\frac{1}{2}\,\Pi\,r'^4 = 115\times10^6\times\frac{6}{3{,}17}\times1{,}57\times\frac{1}{10^8} = 3{,}4\,.$$

Thus the force P at 1m distance from the axis would be 3.4 kg and the arc described at this distance would be $\frac{6\,\Pi}{180} = 0.104\ m$.

167. One can see in *Navier's lessons*, page 106, the results of experiment on the resistance of various substances to torsion.

As for the value T_1 of T' which corresponds to the elastic limit and which it is important not to exceed in constructions and machines, precise experiments are lacking.

Assuming that formula [2] holds up to the instant of rupture, one can, according to known experiments, determine the extreme value of T'. Navier thus found, setting for cylindrical bars (165)

$$P R = \frac{1}{2} T' \Pi r^3$$

that for cast iron the value of T' corresponding to rupture varied from 20×10^6 to 45×10^6 depending on the nature of the cast iron.

It would therefore seem that in practice one could take T up to 4×10^6. But in machines the radius r' calculated from this value would be smaller than the usual dimensions.

Mr. Morin (*Aide-mémoire de mécanique*) sets for cast iron or iron the relation

$$P R = 315.000\, l^3$$

l being the side of the square section. According to this and according to No. 165

$$315.000 = \frac{T}{4{,}24} \quad \text{hence} \qquad T_1 = 1.340\,.000$$

For wood, we can admit that the value of T_1 and that above which suits cast iron are in the same ratio as the values of R_1 belonging to these two substances, or approximately as 0.60 is to 7.50. We thus obtain for wood:

$$T_1 = 0{,}08 \times 1.340\,.000 = 107.000\,.$$

Chap. 4. On friction and the stiffness of ropes.

§: 1 On sliding friction.

168. When two bodies touch each other in such a way as to prevent or modify the movement that would occur by virtue of the forces applied to these bodies, if they were separated, they mutually compress each other. This is a consequence of the physical constitution of bodies, composed of molecules which, in the state of rest or uniform rectilinear motion, are held in equilibrium by the attractive and repulsive forces acting on them; these molecules are consequently ready to yield to any new force introduced into the system. But if a body is hard, the extent of its deformation can be very restricted because a very small change in shape brings into play very large molecular forces.

In *Rational Mechanics*, the existence of these bodies is supposed to be so hard that the contact compression is zero; and from this is drawn the consequence that when two of these bodies touch, the mutual actions manifested at the various points of contact are forces normal to the surfaces at these points; these surfaces are supposed continuous and exactly of the same shape as if the contact did not exist.

169. But this hypothesis is never justified by experiment. If one places a body on another fixed body and abandons it to the action of gravity, however well dressed the two contact faces may be, the plane in which these two faces appear to merge can always be noticeably inclined to the horizon without the upper body sliding on the other, provided this inclination is not too great; from which it follows that the lower body then exerts on the upper one forces whose resultant is oblique to the apparent contact surface, for it is vertical and consequently makes with the apparent contact surface an angle equal to that of this surface with the horizon.

One arrives at the same consequence by observing that a heavy body placed on another whose surface is plane and horizontal does not slide on it, although acted upon by a force parallel to the plane, provided that the ratio of this force to the weight of the body does not exceed a certain limit.

Analogous facts occur during the motion of a body sliding on another. This body, abandoned to the action of gravity on a sufficiently inclined plane, acquires a velocity less than that due to the height of its descent; placed on a horizontal plane and acted upon, in addition to its weight, by a horizontal force, it acquires an acceleration less than that which, considering its mass, should be imparted to it by this force: it is therefore necessary in both cases that the fixed body exert on the moving body oblique reactions whose components parallel to the fixed plane are opposite to the direction of motion.

170. In general, when two bodies touch, an element of the contact surface being considered, if the reaction that one body exerts on the other, over this extent, is oblique to the contact surface, it can be decomposed into two forces, one parallel and the other normal to this

surface: the parallel component is called the sliding friction, and the other the pressure (one should say the normal pressure) of the first body on the second, over the considered extent.

171. The laws of sliding friction are entirely experimental. Numerous observations have shown

1° that when two bodies slide on each other, that is, when the same superficial element of one of them passes successively and continuously into contact with different superficial elements of the other body, each body receives from the other a friction, i.e., a tangential reaction whose direction is opposite to the relative motion of the first body on the second;

2° that the friction for each superficial element in contact is proportional to the intensity of the normal pressure and consequently independent of the extent of the surfaces in contact, when the nature of the bodies does not change, nor their polish, nor their coatings.

3° that friction is independent of the relative velocity of the bodies in contact, at least in ordinary practical cases;

4° that nevertheless, when motion is only on the point of beginning, between two bodies which have been in contact and at rest for some time, the friction is ordinarily more considerable than when motion is established, either by translation or by simple shaking.

172. These laws being admitted, it follows that, calling

R the total oblique reaction that an element of the contact surface, taken on one of the bodies, receives from the other body,

N the pressure over the extent of this element, i.e., the component or projection of R along the normal,

F the friction corresponding to the same element, i.e., the tangential component or projection of R,

α the angle of R with N, called the angle of friction,

f a number called the coefficient of friction which depends on the nature and the state of polish or oiliness of the bodies in contact and which moreover takes only two values according to whether motion is established or only on the point of beginning,

we have $F = fN$, and $F^2 + N^2 = R^2$ hence $F = \frac{f}{\sqrt{1+f^2}} R$;

$$N = \tan\alpha \quad \text{whence} \quad \tan\alpha = f \ ^{(*)}$$

(*) See for the values of f provided by experience, the *introduction to Industrial Mechanics* by *M*. Poncelet or the Practical Mechanics cheat sheet by *M*. Morin.

173. Application to the inclined plane. A body slides on an inclined plane parallel to the line of greatest slope. It is acted upon, in addition to its weight P, by a force Q situated in a plane perpendicular to the horizontals of the inclined plane. The law of motion is required.

Let i be the acute angle of the inclined plane with the horizon, (fig. 40)

β the arbitrary angle of the force Q above the line of greatest slope taken in the ascending direction,

V the velocity, supposed in the ascending direction, at the end of time t.

The translational motion in question is also that of the center of gravity of the sliding body. Transporting all forces P, Q, N, F there, these last two representing the reactions of the inclined plane, we will have a single resultant φ which, being parallel to the direction of the rectilinear motion, will project in its true length onto the line of greatest slope, while its projection onto the normal to the plane will be zero. Thus, given that F acts from top to bottom,

$$\varphi = -P \sin i + Q \cos \beta - F$$
$$N + Q \sin \beta - P \cos i = 0$$

hence, because of the relation $F = f\ N$ (172)

we obtain $$\varphi = Q \cos \beta - P \sin i - f\left(P \cos i_Q \sin \beta\right)$$

a formula easy to find *a priori*.

Consequently we have for the acceleration $\frac{\varphi}{m}$

$$\frac{d\,v}{d\,t} = \frac{g}{P}\left(Q \cos \beta - P \sin i - f\left(P \cos i - Q \sin \beta\right)\right).$$

Therefore if the intensity Q and the angle β remain the same during the motion, the acceleration is constant, and the motion is uniformly varied; according to whether the force φ is positive, negative, or zero, the motion supposed ascending by virtue of the velocity already acquired at the instant considered is accelerated, retarded, or uniform. It is also zero if the body is at rest but on the point of taking an ascending motion.

If, on the contrary, the velocity acquired at the instant considered were descending, it would suffice, while keeping the ascending direction as the positive sense for φ, to change the sign of F in the first of the preceding equations, which amounts to changing that of f in the last formula; and then, according to whether the force φ is positive, negative, or zero, the descending motion will be retarded, accelerated, or on the point of beginning.

Consider the two cases where this body at rest is on the point of starting into motion. If the body is on the point of rising, we have

$$\varphi = 0 \ldots\ldots\ldots\ Q = P\,\frac{\sin i + f \cos i}{\cos \beta + f \sin i}$$

if in this case the force Q is directed below the parallel to the inclined plane, calling β' its angle on that side, it must be that

$$\cos\beta' - f\sin\beta' > 0 \qquad \text{or} \quad \tan\beta' < \frac{1}{f} \quad \text{or} \quad \beta' < 90° - \alpha .$$

If the body is on the point of descending, we have, with change of sign of f, $\varphi = 0$ hence

$$Q = P\,\frac{\sin i - f\cos i}{\cos\beta - f\sin\beta}$$

If Q lies between the two above values, the equilibrium of the body is stable.

If the force Q is zero and the body on the point of descending, we have:

$$\sin i - f\cos i = 0 \qquad \text{or} \qquad \tan i = f$$

Consider now the two possible cases of motion of the body. If it possesses an initial ascending velocity V_0, keeping for φ its previous expression, we have

$$V = V_0 + \frac{g\,\varphi}{P}\,t \qquad \text{and} \qquad x = x_0 + V_0\,t + \frac{g\,\varphi}{P}\,\frac{t^2}{2}$$

and we will thus have the velocity and position of the body after any time t, provided that t is not so large that V becomes negative when φ is negative, for these formulas essentially assume ascending motion.

If, on the contrary, the body possesses an initial descending velocity, counting velocities and x in this same sense, the two last equations subsist under the same condition, but the expression of φ, counted in the descending sense, without changing anything in the definition of angle β, is

$$\varphi = P\sin i - Q\cos\beta - f\,(P\cos i - Q\sin\beta) .$$

§: 2. On the resistance called rolling friction.

174. A body bounded by a curved surface and placed on a horizontal plane with which it is supposed to have only a point or an edge of contact, being acted upon by vertical forces, could not be in equilibrium, if there were no deformation, unless the resultant of these forces passed through the point or edge of contact. Experience teaches that this is not so. Let, for example (fig. 41), a homogeneous cylinder or roller placed on two horizontal planks leaving an interval through which passes a string which, embracing half a turn of the roller, carries at its ends two equal weights $\frac{1}{2}Q$, $\frac{1}{2}Q$: an additional weight q is needed on one side or the other not only for rest to be broken but even for the roller to maintain itself in motion when one has started to shake it. Coulomb found that for a lignum vitae cylinder 0.162 m in diameter rolling on oak rules, when the weights $\frac{1}{2}Q$ including the weight of the roller, totaled 500 kg, the additional weight q had to be 3 kg for a very slow motion to persist after having been imparted. Then the cylinder is in equilibrium under the action of the forces $\frac{1}{2}Q$, $\frac{1}{2}Q + q$, and the resultant of the reactions of the lower body, which is consequently also vertical, equal to $Q + q$, and passing through a point A' in the distance AA' from the vertical diameter of the cylinder, δ, satisfying the equation of moments with respect to A,

$$(Q+q)\,\delta = q\,r \quad \ldots\ldots\ldots [\,1\,] \quad \text{hence} \qquad \delta = \frac{3}{503}\,r = 0.00048\ m\ .$$

These very simple considerations give, as Mr. Dupuit remarked (*Essay on the traction of carriages* . . . . 1837), a clear notion of the fact commonly designated by the name of friction of the second kind or rolling friction, which generally consists in that, when a body rolls or is on the point of rolling on another, the normal components of the reactions of the latter have a resultant which passes at a small distance in front of the normal at the supposed geometric point of contact, on the side of the existing or incipient motion.

175. Coulomb established that the ratio $\frac{q}{Q+q}$ of the additional weight to the normal reaction or pressure is constant for the same radius r, and for the same substances in contact: a fact analogous to one of the laws of sliding friction.

Coulomb believed he recognized moreover that when the radius r varies, the ratio $\frac{q}{Q+q}$ varies inversely, so that we would have

$$\frac{q}{Q+q} = \frac{a}{r}$$

a being a constant. But since, from equation [1] above, we also have $\frac{q}{Q+q} = \frac{\delta}{r}$, it would follow that $\delta = a$, i.e., whatever the radius r, the point A', center of the normal reaction, would be at a constant distance from the normal CA drawn from the axis of the cylinder to the fixed plane. Now it is clear that this law cannot be admitted in general since δ must always be smaller than r.

Mr. Dupuit, in the cited work and in the *Annales des Ponts and Chaussées* (1842 - May), gave the results of experiments from which he concluded that the distance δ, when the substances in contact remain the same, varies proportionally to the square root of the radius; thus one could set

$$\delta = a\sqrt{r} \quad \text{and} \quad \frac{q}{Q+q} = \frac{a}{\sqrt{r}}$$

According to him, we have

	Values	
	of δ	*of* $\frac{q}{Q+q}$
For wood rolling on wood	$0.0011\sqrt{r}$	$\frac{0.0011}{\sqrt{r}}$
For iron —— on moist wood	$0.0010\sqrt{r}$	$\frac{0.0010}{\sqrt{r}}$
For iron —— on iron	$0.0007\sqrt{r}$	$\frac{0.0007}{\sqrt{r}}$
For wheels on paved roads, about	$0.03\ \sqrt{r}$	$\frac{0.03}{\sqrt{r}}$

Mr. Morin, after new experiments, contested the accuracy of this law, and maintained that Coulomb's was closer to the truth in ordinary practical cases.

But Mr. Dupuit persists in his opinion and contradictory experiments alone seem able to remove doubts on the question of fact in question.

176. Suppose now that the cylinder placed on a horizontal plane (fig. 42) and acted upon by vertical forces whose resultant Q' passes through the center of figure C, receives in addition the action of a horizontal force q' at a distance h from the plane, a force sufficient for equilibrium to be on the point of being broken. Then the resultant of Q' and q', and consequently also the resultant of the reactions of the lower body, will be oblique and will pass through a point A' in front of A. The cited authors admit that in this case, the moment $q'h$ of the force parallel to the plane with respect to A or A' is proportional to the normal pressure Q' and has the same value as the moment qr in the preceding case, assuming the pressures $Q + q$ and Q' equal. It would follow that the distance AA' would be the same, for A' being on the resultant of Q' and q', denoting AA' by δ', we must have

$$\delta' = \frac{q'h}{Q'} \quad \text{and consequently } \delta' = \delta$$

because of equation [1] and the hypothesis just stated.

177. In the second case of equilibrium just considered, there is on the part of the plane a tangential reaction equal to q'. Now this reaction must be less than $f Q'$, denoting by f the coefficient of sliding friction; otherwise the cylindrical body, starting from rest, would slide instead of rolling. Therefore for a given value of h, depending on whether the value of q' which determines rolling, namely

$$\frac{\delta Q'}{h} \text{ is } < \text{ or } > f Q'$$

i.e., whether h is $>$ or $<$ $\frac{\delta}{f}$

there will be rolling or sliding on starting from rest. The latter must be very rare.

Let, for example, $\delta = 0.001\sqrt{r}$ and $f = 0.15$: it suffices, for rolling to occur, to have $h > 0.0066\, m\sqrt{r}$.

178. Applications

1st. Let a homogeneous cylinder (fig. 43) whose weight is P, rolling with uniform motion on a horizontal plane: for this purpose it is acted upon by a constant horizontal force Q, supposed applied to a cord wound on a cylinder concentric with the first. The motion being uniform, the external forces acting on the rolling body satisfy the conditions of equilibrium. These forces are P, Q, the normal reaction N passing through A', the tangential reaction F; and we have

$$N = P\ ,\ F = Q\ ,\ Qh = P\,\delta\ .$$

Let us seek the work of Q employed to transport the cylinder a certain distance. Let r be the radius of the cylinder, V the velocity of axis C, and U the velocity of D, point of application of Q; we have $U : V :: h : r$, hence $U = \frac{V h}{r}$. The work of Q per second is therefore

$$Q\,U = \frac{P\,V\,\delta}{r} \qquad \text{hence} \qquad \frac{Q\,U}{P\,V} = \frac{\delta}{r}.$$

The product $P\,V$, of the weight P by the distance traveled in one second, is the useful effect during this time, and we see that the ratio of the work of Q to this effect is independent of the position of the point of application D.

179. 2nd application. A plank, whose weight including its load is P, is placed on two rollers of equal diameter $2r$ which rest on a horizontal plane; we ask what horizontal force Q applied to the plank will maintain the uniform motion of the system. Let (fig. 44) one of the rollers be transported uniformly in the direction indicated by the arrow. This cylinder, considered separately, neglecting its weight, supposed very small compared to P, is in equilibrium under the effect of the reactions it receives from the lower plane and from the plank, and which can be reduced as follows:

1° normal reaction N passing at a distance AA' or δ from the vertical diameter AB.

2° tangential reaction F, which can be considered at will as passing through A or A'.

3° normal reaction N' passing at a distance BB' or δ behind B, since such is the direction of the relative rolling on the lower plane of the plank.

4° tangential reaction F' passing through B or B'.

The equilibrium of these forces requires that $N = N'$, $F = F'$ and, taking moments with respect to A'

$$N'(\delta + \delta') = F'.\,2\,r\,;$$

we would have an analogous result for the second roller

$$N_1'(\delta + \delta') = F_1'.\,2\,r$$

Now the plank is subjected on the one hand to the horizontal force Q and to the downward vertical forces whose sum is P, on the other hand to actions equal and opposite to the vertical forces N', N_1', and to the horizontal forces F', F_1'. We therefore have

$$P = N' + N_1' \qquad \text{and} \qquad Q = F' + F_1'$$

now, adding the two preceding equations, we have

$$(N' + N_1')(\delta + \delta') = (F' + F_1')\,2\,r$$

$$\text{therefore} \ldots\ldots\ldots\ P(\delta + \delta') = 2\,r\,.\,Q$$

$$\text{Example:} \qquad \delta = 0.03\sqrt{r}, \qquad \delta' = 0.001\sqrt{r} \qquad Q = \frac{0{,}031}{2\sqrt{r}}\,P.$$

$$\text{if} \ldots\ldots\ldots\ldots\ r = 0.25, \qquad Q = 0.031\,P.$$

Remark. The velocity of the plank is evidently double that of the axis C' of the roller.

180. 3rd application. A body bounded by a cylindrical surface and having its center of gravity on the axis of figure C (fig. 45) rolls on a plane. It is acted upon, independently of the reactions of this plane, by various forces whose resultant passes through axis C, is perpendicular to it, and can be replaced by two forces, one P normal, the other T parallel to the plane. The law of the rectilinear motion of axis C, and the intensities of the reactions of the plane, are required.

In addition to forces P and T, the cylinder receives from the plane an unknown tangential reaction F, plus a normal reaction N passing through A' and whose intensity is equal to P, since the center of gravity C moves parallel to the plane (38).

Apply to the cylinder the theorem of the effect of work (39).

Let v be the velocity of axis C at any instant. The velocities of the different points of the body are the same as if it were turning about axis A. Let ω be the angular velocity of this rotational movement. The living power of the system at the instant considered is

$$\frac{1}{2}\omega^2 \sum m\, r^2 \qquad \text{or} \qquad \frac{1}{2}\frac{v^2}{R^2}\left(M R^2 + \sum m\, r_1^2\right)$$

calling $\sum m r^2$ the moment of inertia with respect to axis A, M the mass of the system, R the radius AC, $\sum m r_1^2$ the moment of inertia with respect to axis C passing through the center of gravity.

Therefore the increase in living power between two instants where the velocity of C is first v_0 then v, is

$$\frac{1}{2}\left(v^2 - v_0^2\right)\left(M + \frac{\sum m\, r^2}{R^2}\right).$$

Now, in this same interval, the work of T is $T x$, denoting by x the rectilinear path traveled by C parallel to the plane. The work of P is zero. Finally, if we denote by ζ_f the work resulting from the contact of the cylinder and the plane, we have

$$\frac{1}{2}\left(v^2 - v_0^2\right)\left(M + \frac{\sum m\, r^2}{R^2}\right) = T_x - \varsigma_f .$$

According to known experiments, the work ζ_f does not appear to depend on the intensity of the velocity, but only on the pressure P and the radius R. Suppose that, without changing these latter quantities, we reduce T to the value T' necessary to maintain uniform motion; we would have

$$T' x - \varsigma_f = 0 \qquad \text{and at the same time} \quad T' R = P\,\delta$$

from which we conclude

$$\varsigma_f = \frac{P\,\delta}{R}\, x$$

and consequently

$$\frac{1}{2}\left(v^2 - v_0^2\right)\left(M + \frac{\sum m\, r_1^2}{R^2}\right) = \left(T - P\,\frac{\delta}{R}\right) x \quad \ldots\ldots\ldots [\,1\,]$$

Thus the motion of axis C is uniformly varied.

It remains to determine the intensity of F, and it is found by means of the theorem of the motion of the center of gravity. The force F being supposed transported to C, as is the entire mass of the body, we have, applying to this fictitious material point the theorem of the effect of work

$$\frac{1}{2} M \left(v^2 - v_0^2\right) = \left(T - F\right) x \ .$$

From this equation we obtain

$$F = \frac{T \sum m\, r_1^2 + P\, M\, R\, \delta}{\sum m\, r_1^2 + M\, R^2} \quad \ldots\ldots\ldots [\,2\,]$$

Example. If it concerned a homogeneous cylinder abandoned to the action of gravity, on an inclined plane making an angle i with the horizon, we would have

$$T = M\ g \sin i\ , \qquad P = M\ g \cos i\ , \qquad \sum m\, r_1^2 = \frac{1}{2} M\ R^2\ .$$

Equation [1] would become

$$v^2 - v_0^2 = \frac{4}{3} g\ x \left(\sin i - \frac{\delta}{R} \cos i\right) \quad \ldots\ldots [\,3\,]$$

while if one admitted the hypothesis of a body sliding without friction on the same inclined plane, one would have

$$v^2 - v_0^2 = 2\, g\ x \sin i\ .$$

Equation [2], by the same substitution, becomes

$$F = \frac{1}{3} M\ g \left(\sin i + 2 \cos i\ \frac{\delta}{R}\right).$$

Moreover, the force F cannot exceed a certain limit, since the cylinder does not slide on the plane. We therefore have

$$F \ < \ f N \qquad \text{or} \qquad F \ < \ f\, M\, g \cos i$$

$$\text{so} \qquad \frac{1}{3} M\ g \left(\sin i + 2\,\frac{\delta}{R} \cos i\right) \ < \ f\ M\ g \cos i$$

$$\text{or} \qquad \tan i \ < \ 3 f - 2\,\frac{\delta}{R}\ .$$

Therefore if the inclination i is three times the angle of friction, there is sliding and equation [1] is no longer applicable.

Mr. Dupuit in his essay arrived by another method at the same formulas.

181. *Particular case*. Homogeneous cylinder rolling upwards under the action of gravity and the reactions of the inclined plane.

Formula [3] subsists by changing the sign of x. If we suppose for greater simplicity that $\frac{\delta}{R}$ can be neglected compared to sin i and that the final velocity v is zero, we have the height of the ascent

$$x \sin i = \frac{3}{2} \frac{v_0^2}{2g} .$$

This is because the living power of the cylinder at the initial instant is $\frac{3}{2} . \frac{1}{2} M v_0^2$.

It will be noted that in this case the force F, the tangential component of the reaction of the plane, and the necessary cause of the slowing down of the rotational motion, acts in the ascending direction, because the cylinder tends to slide in the direction of descent while its axis rises.

182. In a great number of questions of practical mechanics, rolling friction can be neglected. Such is the case of a lever, or of a balance beam, which rests on its support by a knife-edge and not by a trunnion.

§: 3. Wedge press.

183. In the apparatus represented by figure 46, the relation between the force P exerted on the head of the wedge and the pressure Q received by the material being pressed is required.

The wedge, at the instant it is on the point of descending, is subjected to the five forces P, N, N', fN, fN', and if it is supposed isosceles, we have

$$[\,1\,] \ldots\ldots\ldots\ldots\ldots \quad P = 2N(\cos\alpha + f \sin\alpha)$$

The block interposed between the wedge and the material to be pressed is subjected, neglecting its weight, to five forces N_1, fN_1, N'', fN'', Q, the first two of which are equal and opposite to N and fN, by virtue of the principle of equality between action and reaction.

The equilibrium of this block gives two equations whose terms depend only on the intensities and direction of the forces, whatever their unknown points of application:

$$N \sin\alpha - fN\cos\alpha - fN'' - Q = 0$$
$$N\cos\alpha + fN\sin\alpha - N'' = 0$$

hence, eliminating N''

$$[\,2\,] \ldots\ldots\ldots\ldots \quad N(\sin\alpha - 2f\cos\alpha - f^2\sin\alpha) = Q$$

from [1] and [2] we obtain

$$[\,3\,] \ldots\ldots \quad P = \frac{2(\cos\alpha + f\sin\alpha)}{\sin\alpha - 2f\cos\alpha - f^2\sin\alpha} Q = \frac{2(1 + f\tan\alpha)}{\tan\alpha - 2f - f^2\tan\alpha} Q$$

Example. $\tan\alpha = 20$; $f = 0.10$; $\frac{P}{Q} = 0.306$; $\frac{Q}{P} = 3.27$;

Thus while if friction were zero, we would have $\frac{P}{Q} = 0.1$. The limit of smallness of the ratio $\frac{P}{Q}$ corresponds to *tan* $\alpha = \infty$, which is seen by putting equation [3] in the form

$$\frac{P}{Q} = \frac{2\left(f + \dfrac{1}{\tan\alpha}\right)}{1 - f^2 \dfrac{2f}{\tan\alpha}}$$

it is therefore $\frac{2f}{1-f^2}$; while if one neglected friction, $\frac{P}{Q}$ would approach zero indefinitely.

184. Equilibrium being established as before between the forces *P, Q* and the frictions, suppose that an additional force has imparted a movement to the system and then ceases, abandoning it to the forces *P* and *Q*; in the case where these forces then remained constant, the motion would be uniform, and one can propose to compare in this hypothesis the work of *P* to that of *Q*.

If one denotes by x (fig. 47) the vertical path described by the head of the wedge and by ζ the horizontal path traversed at the same time by the movable block of the press, *Px* is the work of P and $Q\varsigma$ that of the pressure applied to the compressed material.

Now we have

$$[\,4\,] \ldots\ldots \quad x = \frac{1}{2}\varsigma \tan\alpha$$

and multiplying [3] and [4] we obtain

$$P\,x = \frac{\tan\alpha + f\tan^2\alpha}{\tan\alpha + 2f - f^2\tan\alpha} Q\varsigma$$

or

$$P\,x = \left(1 + \frac{f\left(\tan^2\alpha + 2 + f\tan\alpha\right)}{\tan\alpha - 2f - f^2\tan\alpha}\right) Q\varsigma\,.$$

One could have concluded this relation immediately from equation [3] by noting that the ratio of the paths x , ς is such that $P\,x$ and $Q\varsigma$ become equal when *f* becomes zero.

The work $Q\varsigma$ is the useful effect and is found to be much lower than the motive work $P\,x$. The loss is even more considerable when the force P is produced by a shock; the deformation of the parts of the apparatus, that of the supports, and their shaking cause resistant works, and a consumption of power, to the detriment of useful work. These losses are diminished, for an equal expenditure of work, when one diminishes the height *h* of fall of the striking body, increasing its mass so that the product *mh* remains the same.

§: 4. On the windlass.

185. The weight of the windlass (fig. 48) is *Π*; two forces P and Q act upon it in addition, in planes perpendicular to the axis. Motion is on the point of beginning, and it is supposed that this occurs in the direction determined by force *Q,* whose work would be positive and

which is consequently the moving force. The aim is to determine Q, knowing Π, P, the dimensions of the work, the directions of P and Q.

One can, without changing the conditions of equilibrium, replace each of the forces Π, Q, P by two other parallel forces acting in planes always perpendicular to the axis, but passing through the bearings A' and A''.

The forces Π' and P' being known, can be combined into a single force S' (fig. 49).

At the point of contact B' of the trunnion and the bearing, a reaction occurs whose components, one normal and the other tangential, are N' and fN'.

Thus in the plane of bearing A' will act the four forces Q', S', N', fN'; and in the plane of bearing A'' will act the four analogous forces Q'', S'', N'', fN''.

For each of these two systems of forces in particular, the resultant of translation must be separately zero; for in the opposite case, the windlass subjected to two or three forces acting in parallel planes could not remain in equilibrium. If we call R' the resultant of forces Q' and S', it must therefore be equal and opposite to the resultant of forces N' and fN'; and since these latter are mutually perpendicular, we must have

$$R'^2 = N'^2 + f^2 N'^2$$

hence $fN' = \frac{fR'}{\sqrt{1+f^2}} = f_1 R'$, replacing, for brevity, $\frac{f}{\sqrt{1+f^2}}$ by f_1, which moreover differs very little from f.

Similarly, if R'' is the resultant of the two forces Q'' and S'', we will have

$$fN'' = \frac{fR''}{\sqrt{1+f^2}} = f_1 R''$$

Now, the windlass being in equilibrium under the action of the forces Π, P, Q, N', N'', fN', fN'', we have, taking moments with respect to the axis of rotation

$$Qq = Pp + fN'r' + fN''r''$$

or

$$Qq = Pp + f_1 R'r' + f_1 R''r'' \quad \ldots\ldots\ldots\ldots [3]$$

(q, p, r' and r'' denoting the radius of the wheel, of the shaft and of the two trunnions)

The problem is thus put into equations, because the forces Q' and Q'', which enter into the values of R' and R'', and consequently into those $f_1 R'$ and $f_1 R''$, have with the force Q determined ratios depending solely on the distances from the plane of the wheel to the planes of the bearings.

Assuming that the forces Q and P keep their directions during the motion of the windlass, we will obtain the work of Q for each complete turn of the windlass by multiplying Qq by 2π. Namely

$$Q \,.\, 2\,\Pi\, q = P \,.\, 2\,\Pi\, p + f_1 R' 2\,\Pi\, r' + f_1 R'' \,.\, 2\,\Pi\, r''$$

$P \,.\, 2\,\Pi\, p$ is the dynamic effect; the remainder of the second member is the work consumed by friction.

186. *1st example*. The forces P and Q are vertical (fig. 50) and denoting by H and K the points where the planes perpendicular to the axis passing through the forces P and Q meet this axis, if we set

$$A'A'' = a\,, \qquad A'H = m'a\,, \qquad A'G = l'a\,, \qquad A'K = n'a\,.$$
$$A''H = m''a\,, \qquad A''G = l''a\,, \qquad A''K = n''a\,.$$

It follows that

$$P' = m''P\,, \qquad \Pi = l''\Pi\,, \qquad Q' = n''Q$$
$$P'' = m'P\,, \qquad \Pi'' = l'\Pi\,, \qquad Q'' = n'Q$$

then $R' = m''p + l''\Pi + n''Q\,;$ $R'' = m'P + l'\Pi + n'Q$

and equation [3] which serves to determine Q, is then of the first degree.

Noting that $m' + m'' = l' + l'' = n' + n'' = 1$, we find, in the particular case where $r' = r''$,

$$Q = \frac{Pp + f_1(P + \Pi)\,.\,r'}{q - f_1 r'}$$

formula independent of the distance a and its subdivisions at points H, G, K.

187. *2nd example*. The forces P and Q are still vertical, but one of them, P, acts beyond the trunnion A'. (fig. 51)

Let always $A'A'' = a$

$$A'H = m'a\,, \qquad A'G = l'a\,, \qquad A'K = n'a$$
$$A''H = m''a\,, \qquad A''G = l''a\,, \qquad A''K = n''a$$

We deduce as above

$$P' = m''P\,, \qquad \Pi = l''\Pi\,, \qquad Q' = n''Q$$
$$P'' = m'P\,, \qquad \Pi = l'\Pi\,, \qquad Q'' = n'Q$$

then $R' = m''P + l''\Pi + n''Q$

$$R'' = -m'P + l'\Pi + n'Q$$

and $Qq = Pp + f_1 R'\,.\,r' = f_1 R''r''$

The terms $f_1 R'\,.\,r'$ and $f_1 R''\,.\,r''$ being essentially positive, since the friction of the trunnions is always exerted in the opposite direction to the motion of the windlass, whether they touch the lower or upper part of the bearing, only the absolute value of R'' should be taken.

In ordinary cases where Qq differs little from Pp, and where consequently $\frac{Pp}{q}$ is an approximate value of Q, one can ascertain in advance whether one should take

$$R'' = -m'P + l'\Pi + n'Q \qquad \text{or} \qquad R'' = m'P - l'\Pi - n'Q$$

in all cases one can verify after calculating Q whether one has started from a false hypothesis.

88. *3rd example*. The force P is vertical like Π, but the force Q is horizontal; all these forces act between the planes passing through A' and A''. In this case, we have

$$S' = P' + \Pi', \qquad S'' = P'' + \Pi'', \qquad R' = \sqrt{S'^2 + n''^2 Q^2}, \qquad R'' = \sqrt{S''^2 + n'^2 Q^2}$$
$$Q\,q = P\,p + f_1\,r'\,R' + f_1\,r''\,R''.$$

R' and R'' are functions of the known forces S' and S'', and of the unknown Q. Eliminating R' and R'' would lead to an equation in Q of the 4th degree. It is better in numerical application cases to proceed by trial and error for the determination of Q. We note that, given the smallness of the coefficient f_1 and of the radii r', r'' compared to q and p, the product Qq is only slightly greater than Pp, from which we conclude an approximate value of Q as a number for which it will suffice to take the first two significant digits. Substituting this quantity into the expressions for R' and R'', we obtain numerically their approximate values and by transferring them into the last formula above, we obtain, with generally sufficient approximation, the value of Q. If, however, this differed noticeably from the first hypothesis, it could be carried back into the formulas for R' and R'' and consequently lead to a more accurate evaluation of Q.

It may happen that an algebraic expression of Q is needed, no longer containing the auxiliary results R', R''. This is achieved simply by means of an approximate formula due to Mr. Poncelet, as we shall now explain.

189. Suppose we wish to replace approximately the radical $\sqrt{x^2+y^2}$ by the binomial $\alpha x + \beta y$; one could choose α and β so that for two designated values of $\frac{x}{y}$ there is equality between these two expressions. But for any other hypothesis there will be a difference, and the ratio of this difference to the true value $\sqrt{x^2+y^2}$ will be the relative error.

Let us represent it by z.

We will have $\quad z = \dfrac{\alpha x + \beta y - \sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} - 1 \quad$ or $\quad z = \dfrac{\alpha n + \beta}{\sqrt{n^2+1}} - 1$,

denoting the variable $\frac{x}{y}$ by n.

Suppose we know that the ratio n can only vary between the number K and infinity. (if for example we know that $x > y$, it follows that $K = 1$)

The problem is to determine α and β so that, between $n = K$ and $n = \infty$, the greatest absolute value of z is a minimum with respect to the various values one could choose for α and β.

for $\quad n = K \quad$ we have $\quad z' = \dfrac{\alpha K + \beta}{\sqrt{K^2+1}} - 1$

and for $\quad n = \infty \quad$ $z'' = \alpha - 1$

Between these two values, z considered as a function of n is susceptible to a maximum which will be determined by differentiation. We find

$$\frac{dz}{dn} = \frac{\alpha - \beta n}{(n^2+1)\frac{3}{2}}.$$

Hence we conclude 1° that $\frac{dz}{dn}$ decreases as n increases, 2° that the maximum of z corresponds to $n = \frac{\alpha}{\beta}$.

The maximum is . . . . . $z''' = \sqrt{\alpha^2+\beta^2} - 1$

That established, Mr. Poncelet remarks that one could determine α and β so that the extreme errors z' and z'' were zero, which would give

$$\alpha = 1 \qquad \beta = \sqrt{K^2+1-1}$$

but that it is better, by decreasing α and β, to make z' and z'' negative and thereby make the maximum z''' smaller. This leads him to set the double condition

$$Z''' = -z' \qquad \text{and} \qquad z''' = -z''$$

which ensures that in all cases the numerical value of the positive or negative error will not exceed z''' or $\alpha - 1$.

We therefore have two equations to calculate α and β when K is given, namely

$$\alpha^2+\beta^2 = 2 - \frac{\alpha K + \beta}{\sqrt{K^2+1}} = 2 - \alpha$$

for $K=0$ we obtain $\alpha = \beta = 0.83$ and $z''' = 0.17$ or $\frac{1}{6}$.

for $K=1$ we find $\alpha = 0.96$, $\beta = 0.40$ and $z''' = 0.04$ or $\frac{1}{25}$.

Thus when we know in advance that x cannot be smaller than y, we can set approximately

$$\sqrt{x^2+y^2} = 0.96\,x + 0.40\,y$$

and be assured that the error will not exceed $\frac{1}{25}$ of the exact value.(*)

(*) We find in the lithographed lessons of M. Poncelet at the school of application of Metz a table of the values of α and β and the error limit $1-\alpha$, calculated for the values of K from 0 to 10.
We can find in the following way the formulas that he gave to facilitate these calculations.
Let be the error limit corresponding to K. We have the three equations

$$\epsilon = 1-\alpha \qquad \epsilon = \sqrt{\alpha^2+\beta^2} - 1 \qquad \epsilon = 1 - \frac{\alpha K+\beta}{\sqrt{1+K^2}}$$

The first two give $\alpha = 1-\epsilon$ et $\beta = 2\sqrt{2}$ and the 3rd becomes by substitution

$$1-\epsilon = \frac{(1-\epsilon)K + 2\sqrt{2}}{\sqrt{1+K^2}} \quad \text{or} \quad K + \frac{2\sqrt{2}}{1-\epsilon} = \sqrt{K^2+1}$$

Either to abbreviate $\frac{2\sqrt{2}}{1-\epsilon} = \epsilon'$; it follows from this

$$K + \epsilon' = \sqrt{K^2+1} \quad \text{hence} \quad 2\epsilon' K + \epsilon'^2 = 1 \quad \text{hence} \quad \frac{1}{K} = \frac{2\epsilon'}{1-\epsilon'^2}$$

The $\frac{2\sqrt{2}}{1-\epsilon}$ and $\frac{2\epsilon'}{\sqrt{1-\epsilon'^2}}$ expressions lead to thinking

$\epsilon = \tan^2 \varphi$ hence $\epsilon' = \tan 2\varphi$ and $\frac{1}{K} = \tan 4\varphi$

at the same time as $\alpha = 1 - \tan^2 \varphi$ and $\beta = 2 \tan \varphi$.

From this, given *K*, we will conclude the angle of φ then the values of ϵ , of α and of β.
Here is the table calculated in Metz by Mr. Gosselin

Values of K	α	β	$1 - \alpha = \epsilon$ error limit	Values of K	α	β	$1 - \alpha = \epsilon$ error limit
0	0.8281	0.8284	1/6	6	0.9983	0.0826	1/589
1	0.9605	0.3978	1/25	7	0.9987	0.0710	1/800
2	0.9859	0.2327	1/71	8	0.9991	0.0622	1/1089
3	0.9935	0.1612	1/154	9	0.9993	0.0554	1/1428
4	0.9962	0.1226	1/266	10	0.99935	0.0498	1/1538
5	0.9976	0.0988	1/417				

190. Let us apply this theorem to the example of No. 188. If we are sure that S' and S'' exceed $n''Q$ and $n'Q$, we will have, to within $\frac{1}{25}$,

$$R' = 0{,}96\, S' + 0.40\, n''Q \qquad \text{and} \qquad R'' = 0.96\, S'' + 0.40\, n'Q \ ;$$

then, substituting in the expression for Qq and solving,

$$Q = \frac{P\,p + 0.96\, f_1 (r'S' + r''S'')}{q - 0.40\, f_1 (r'n'' + r''n')} \ .$$

191. It may happen that the windlass is acted upon by a force F acting in the direction of the axis; there then arises an equal and opposite reaction, either against the end of one of the trunnions, or against its shoulder. In either case, the moment of the frictions which occur must be added to the second member of equation [3] of No. 185.

In the first case, let A be the axis of the circle which terminates the trunnion; admit as sufficiently approximate that the pressure F is uniformly distributed over this surface; $\frac{F}{A}$ will be the pressure per unit area. If a denotes a superficial element of the circle, and x its distance from the center, $\frac{F}{A} . a$ will be the pressure supported by this element, $f \frac{F}{A} . a$ will be the corresponding friction, and $f \frac{F}{A} . a\, x$ the moment of this friction. The quantity to be introduced into the second member of equation [3] of No. 185 will therefore be $\sum f \frac{F}{A} a\, x$ or $f \frac{F}{A} \sum a\, x$; this sum extending over all elements of the circle, whose radius we will denote by ζ.

The part of $\sum a\, x$ which would correspond to an annular zone of width dx would reduce to $2\, \Pi\, x^2\, d\, x$.

Therefore the entire sum is

$$\int_0^{\zeta} 2\,\Pi\, x^2\, dx \quad \text{or} \quad \frac{2}{3}\,\Pi\,\zeta^3 \quad \text{or} \quad \frac{2}{3}\,A\,\zeta$$

hence $f\,\frac{F}{A}\,\Sigma\, a\, x$ becomes $f\,F\,\frac{2}{3}\,\zeta$ and the moment equation is

$$Q\,q = P\,p + f\,N'\,r' + f\,N''\,r'' + f\,F\,\frac{2}{3}\,\zeta\,.$$

We see that ζ must be reduced as much as possible, which is achieved in several ways (fig. 52 and 53); but there is a limit beyond which the pressure $\frac{F}{A}$ per unit area becoming too considerable, the materials in contact would run the risk of being damaged.

192. If the pressure F, or rather the reaction equal and opposite to it, bears against the shoulder, F/A will still be the pressure per unit area. Let ζ and ζ' be the radii of the inner and outer circumferences of the ring. We will evidently have the moment of frictions, by calculating them as above for radius ζ', then for radius ζ, and subtracting the first result from the second; we find

$$\frac{2}{3}\,\frac{f\,F\,\Pi\,(\zeta^3 - \zeta'^3)}{A}$$

But $A = \Pi\,\zeta^2 - \Pi\,\zeta'^3$; the moment of frictions therefore becomes

$$\frac{2}{3}\,\frac{f\,F\,\Pi\,(\zeta^3 - \zeta'^3)}{\Pi\,(\zeta^2 - \zeta'^3)} \qquad \text{or} \qquad \frac{2}{3}\,\frac{f\,F\,(\zeta^2 + \zeta\,\zeta' + \zeta'^2)}{\zeta + \zeta'}$$

If we call ζ_1 the mean radius, and l the width of the ring, we have

$$\zeta + \zeta' = 2\,\zeta \qquad \text{and} \qquad \zeta - \zeta' = l\,;$$

and from the values of ζ' and ζ derived from these relations, the above expression becomes

$$f\,F\left(\zeta_1 + \frac{1}{12}\,\frac{l^2}{\zeta_1}\right)$$

If, besides the forces $\Pi\,P\,Q$ of No. 185, there is a force F (fig. 55) parallel to the axis, and at a distance d from this axis, we can, without changing the reactions of the supports, introduce two forces F_1 and F_2 equal to each other and to the force F, and acting in opposite directions along the axis; then replace the couple $F_1\,F_2$ by an equivalent couple F', F'', situated in the same plane. The forces F' and F'' must, for equivalence of couples, satisfy the relation

$$F'\,.\,\overline{A'A''} = F\,d\,.$$

We then have three systems of forces, namely

1° in the plane perpendicular to the axis, at A'. . . . . Π', P', Q', F', N', $f\,N'$; plus the frictions at the end of the trunnion or on the shoulder.

2° in the plane perpendicular to the axis, at A''. . . . . Π'', P'', Q'', F'', N'', $f\,N''$;

3° along the axis F_1, equal to F, and an equal and opposite reaction.

The resultant of this last system being zero, it is necessary, as in No. 105, that the resultant of translation of each of the other two systems be separately zero, and independently of the friction on the end or on the shoulder, which reduces to a couple;

Calling S' and S'' the resultants of translation of the forces Π', P', F' on the one hand and Π'', P'', F'' on the other, we will obtain two equations similar to [1] and [2] of No. 185. Then, according to whether the translation of the windlass in the direction of its axis is prevented by a plane bearing against the end of the trunnion or against its shoulder, we will have

$$\left.\begin{aligned} Qq &= Pp + f_1 r' \sqrt{R'^2} + f_1 r'' \sqrt{R''^2} + \frac{2}{3} f \zeta F \\ \text{or} \qquad Qq &= Pp + f_1 r' \sqrt{R'^2} + f_1 r'' \sqrt{R''^2} + fF\left(\zeta_1 + \frac{1}{12}\frac{l^2}{\zeta_1}\right) \end{aligned}\right\} \quad \ldots\ldots [\,3\,]$$

Equations [1], [2], [3] provide the means to determine the intensity of Q when its direction is known. Instead of proceeding by elimination, it will be more convenient to take first for Q the approximate value $\frac{P\,p}{q}$ resulting from the hypothesis where friction would be zero, to substitute this value or a slightly larger number in [1] and [2]; we will conclude from them approximate values of $\sqrt{R'^2}$ and $\sqrt{R''^2}$, which, introduced into [3], will give a sufficiently approximate value of Q. If a greater approximation were desired, this last value would be carried back into [1] and [2]; the values of $\sqrt{R'^2}$ and $\sqrt{R''^2}$ derived from them, substituted in [3], would give a value of Q very little different from the preceding one and from the true one.

One could also make use of Mr. Poncelet's theorem (189); for this purpose, all forces P', F', Q', etc., are replaced by their horizontal and vertical components; then the resultants R', R'' exactly of the form $\sqrt{x'^2 + y'^2}$, $\sqrt{x''^2 + y''^2}$ will be reduced approximately to $\alpha x' + \beta y'$ and to $\alpha x'' + \beta y''$; and equation [3] will finally reduce to the first degree in Q.

194.(*) The direction of Q being always given, we may have to determine its intensity for the case where the motion would take place in the opposite direction to that of this force, P then being the moving force, and Q the resisting force. Denote by Q_1 the value of Q for this case, by R'_1 and R''_1 the corresponding values of R' and R''; equations [1], [2], [4] will subsist by changing Q, R', R'' into Q_1, R'_1, R''_1; but instead of equation [3] we will have

$$\left.\begin{aligned} Q_1 q &= P p - f_1 r' \sqrt{R'^2_1} - f_1 r'' . \sqrt{R''^2_1} - \frac{2}{3} f \zeta F \\ \text{or} \qquad Q_1 q &= P p - f_1 r' \sqrt{R'^2_1} - f_1 r'' \sqrt{R''^2_1} - fF\left(\zeta_1 + \frac{1}{2.}\frac{l^2}{\zeta_1}\right) \end{aligned}\right\} \quad \ldots\ldots [\,3\text{ bis}\,]$$

(*) Here, there is no N° 193. A new 192, followed by the 193 and a new 194 (with different contents), follow directly.

in ordinary cases, where the quantities f, r', r'' are small, R' and R'' differ little from R'_1 and R''_1, and we can, by subtracting [3 bis] from [3], write

$$\frac{1}{2}\left(Q - Q_1\right) q = \text{the moment of friction}$$

Hence a means of determining this moment by experiment: it suffices to observe the smallest force Q necessary, with a lever arm q, to turn the windlass in one direction, and the smallest force Q_1 necessary to prevent it from turning in the opposite direction, then to take half the product $\frac{1}{2}(Q - Q_1) q$.

192. If we suppose the windlass animated with a uniform motion, the relations established above will still hold. Calling ω the angular velocity, the arc traversed per unit time by each point of the circumference which has radius p will be $p\,\omega$; the work of the moving force and that of the resistance per unit time will therefore have the expression $Q \,.\, q\,\omega$ and $P \,.\, p\,\omega$. Thus, multiplying both members of equation [3] by ω, we see that the motive work is equivalent to the work of the resistance P, plus the work of friction, which the preceding formulas provide the means of calculating.

193. If the articulations of balances were formed by trunnions turning in bearings of approximately equal diameter, the preceding theory would be applicable to them. But these articulations being made by knife-edges, friction no longer occurs with sliding, but only with rolling, and becomes very slight if the bodies in contact are hard and polished.

194. Roller trunnions. *Trunnion A (fig. 56) carrying a load P rests on two rollers or wheels, which can turn on their trunnions B and B′.* To treat only a very simple case, it is supposed that the direction of P, passing through center A, at equal distance from the planes of the two wheels, divides the angle BAB' into two equal parts. The force Q which, acting at distance q from axis A, would be on the point of making the main body rotate about trunnion A, is required.

If the force Q did not act, the wheels would receive and each exert, at points C and C', the same pressure N, and calling 2α the angle BAB', we would have

$$P = 2\,N \cos\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots [\,1\,]$$

The force Q being expected to be small, we will admit for simplicity that its action does not appreciably alter the pressures N; moreover, we will neglect rolling friction, i.e., we will suppose that the pressures N continue to pass through A.

That established, let r, a, ρ be the radii of trunnion A, of the wheels, and of their trunnions B and B'; let φ be the tangential resistance opposed by each wheel; we will have the moment equation

$$Q\,q = 2\,\varphi\,r \quad \ldots\ldots\ldots\ldots\ldots\ldots [\,2\,]$$

Wheel B is in equilibrium under the action of a force tangential to its circumference, equal and opposite to force φ, of a normal force N along CB, and of the reaction of the bearing on its trunnion. Therefore, calling R the resultant of forces φ and N, we have

$$R^2 = \varphi^2 + N^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots [\,3\,]$$

and

$$\varphi a = \frac{f}{\sqrt{1+f^2}} R\,\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots [\,4\,]$$

these two equations, together with the two preceding ones [1] and [2], contain the solution of the problem. To obtain it approximately, we note, from [4], that φ is very small compared to R, that consequently, from [3], R is sensibly equal to N, thus equation [4] becomes approximately $\varphi a = \frac{f}{\sqrt{1+f^2}} N\rho$, and substituting the values of N and φ derived from [1] and [2], we obtain

$$Q\,q = \frac{f\,P\,r}{\sqrt{1+f^2}} \cdot \frac{\rho}{a} \cdot \frac{1}{\cos\alpha}$$

while if trunnion A turned in a fixed bearing, we would have, assuming q very large compared to r,

$$Q\,q = \frac{f\,P\,r}{\sqrt{1+f^2}}$$

The moment $Q\,q$ of the power must therefore be smaller, the larger a is compared to ρ, and the smaller the angle α.

The use of this apparatus supposes that the wheels remain exactly cylindrical, which is difficult under great pressures; and when the wheels cease to turn, the advantages of this device become illusory.

§: 5. Formulas relating to the traction of carriages.

195. To simplify, consider a single wheel which rolls on horizontal ground and whose hub is traversed by the axle to which the rest of the vehicle is invarably linked. Let

Q be the resultant of the vertical forces applied to the vehicle, the wheel not included (fig. 57 and 58),

q the distance of the resultant Q behind the vertical line passing through the axle center,

T the traction, i.e., the horizontal force applied to the vehicle to maintain uniform motion,

h the height of the horizontal line of T above the axle center,

P the weight of the wheel, whose center of gravity is on the axis of the wheel, or of the hub box,

r the radius of the wheel,

ρ the radius of the hub box,

ρ' the radius of the axle, slightly smaller than ρ.

The wheel receives from the ground various reactions which can be reduced to two forces, one horizontal equal and opposite to T, which we will call T', the other vertical equal to $Q + P$,

which we will designate by $Q' + P'$. This results from the theorem of the motion of the center of gravity of any material system. The force $Q' + P'$ can be considered as applied to the wheel at a point A', at a distance δ in front of the vertical line OA passing through the axis of the wheel or hub.

That established, consider separately the forces acting on the wheel, which satisfy the conditions of equilibrium. These forces are T', $Q' + P'$, P, and the actions of the axle on the concave surface of the hub box, actions which can be reduced to two forces, one normal N, the other tangential $f N$ directed opposite to the direction of rotation of the wheel (whether the contact edge of the axle and the hub is found in front of or behind the vertical OA; we will distinguish these two cases later).

The five forces T', $Q' + P'$, P, N, $f N$ have zero algebraic sum of their moments with respect to any axis. Take for this axis that of the wheel or hub; the moments of P and N are zero and by putting for T' and $Q' + P'$ their values T and $Q + P$, we have

$$T r - (Q + P)\,\delta - f N \rho = 0 \quad \ldots\ldots\ldots [\,1\,]$$

Moreover, the resultant of the forces N and $f N$, which are perpendicular, is equal to that of the forces T, $Q' + P'$, and P, which, given that P and P' are equal and of opposite directions gives the relation

$$N^2 (1 + f^2) = T^2 + Q^2 \quad \ldots\ldots\ldots\ldots [\,2\,]$$

Now the forces acting on the vehicle, without including the wheel, are Q, T and the reactions which we will designate by N' and $f N'$, which are equal and opposite to the forces N and $f N$ considered above. Their moments taken with respect to the axis O' of the axle give the equation

$$Q q - T h - f N \rho' = 0 \quad \ldots\ldots\ldots\ldots [\,3\,]$$

The equilibrium of the same forces Q, T, N' and $f N'$ would also lead to equation [2] already obtained.

196. By means of the three equations above, if the quantities Q, h, r, ρ', δ, f are supposed given, we will determine T, q and the auxiliary unknown N. From [2] we obtain

$$f N = \frac{f}{\sqrt{1 + f^2}} \sqrt{T^2 + Q^2} \qquad \text{or} \qquad f N = f_1 \sqrt{T^2 + Q^2}$$

and substituting in [1]

$$T = (Q + P)\frac{\delta}{r} + \frac{f_1 \rho}{r} \sqrt{T^2 + Q^2} \quad \ldots\ldots\ldots [\,4\,]$$

formula in which $\sqrt{T^2 + Q^2}$ can be approximately replaced by Q, since T is ordinarily small compared to Q.

197. Equation [3] by substituting the value of $f N$ becomes

$$Q q = T h + f_1 \rho' \sqrt{T^2 + Q^2}\,,$$

from which we see that q cannot be zero or negative unless h is negative.

198. Let us now determine the situation of the edge B of contact of the axle and the hub. Suppose it is in front of the vertical OA, (fig. 57) and call α the angle AOB.

The equilibrium of forces Q, T, N', $f\,N'$ gives two equations between their projections, namely

$$\left.\begin{aligned} Q - N\cos\alpha + f\,N\sin\alpha = 0 \\ T - f\,N\cos\alpha - N\sin\alpha = 0 \end{aligned}\right\} \quad \text{hence} \quad \sin\alpha = \frac{T - f\,Q}{\left(1+f^2\right)N} \quad \text{and} \quad \cos\alpha = \frac{Q + f\,T}{\left(1+f^2\right)N}$$

If B were behind the vertical OA (fig. 58), calling α the angle AOB, we would find two equations similar to the preceding ones, except that the terms containing sin α would have their signs changed. Therefore the last two formulas are applicable to both cases, provided it is understood that if sin α becomes negative, the angle α is also negative and point B is behind the vertical OA. This occurs when δ is sufficiently small: let us take the extreme hypothesis $\delta = 0$; formula [4] gives $T = \frac{f_1\rho}{r}Q$, a quantity smaller than $f\,Q$.

199. We have previously made known the uncertainties relating to the determination of the quantity δ, which according to Coulomb and Mr. Morin would be almost independent of the radius r, while according to Mr. Dupuit it would be proportional to $\sqrt{r}$. We refer to the writings of this author on this matter.

200. Suppose now that the ground makes an angle i with the horizon, and that the motion being ascending, the force T makes an angle α above the parallel to the ground.

The tangential reaction at the ground opposite to the motion being denoted by T_1, we have

$$T_1 = T\cos\alpha - (Q+P)\sin i$$

and its normal reaction N_1 has the expression

$$N_1 = (Q+P)\cos i - T\sin\alpha\,.$$

That established, we will find two equations similar to formulas [1] and [2] of No. 195, namely,

1° the moment equation

$$T_1 r - N_1\delta - f\,N\rho = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots [\,1\,]$$

2° the equation expressing that the resultant of N_1 and $f\,N_1$ is equal to that of the forces T_1 and Q making an angle $\frac{\pi}{2} + \alpha + i$

$$N^2\left(1+f^2\right) = T^2 + Q^2 - 2\,T\,Q\sin\left(\alpha + i\right) \;\ldots\ldots\ldots [\,2\,]$$

Substituting the values of T_1 and N_1 then eliminating N leads to the final relation

$$T\cos\alpha = (Q+P)\sin i + \left((Q+P)\cos i - T\sin\alpha\right)\frac{\delta}{r} + \frac{f_1\rho}{r}\sqrt{T^2 + Q^2 - 2\,T\,Q\sin(\alpha+2)}$$

Mr. Morin in his memoir on the traction of carriages calls the resistance opposed by the ground to rolling, referred to the wheel circumference, a force which he denotes by R and which, according to the formula he gives, is the quantity

$$\left((Q+P)\cos i - T\sin\alpha\right)\frac{\delta}{r}$$

entering into the above equation. In applications he takes $f_1 = 0.065$.

§: 6. On the square-threaded screw.

201. It is supposed that the nut (fig. 69) is immobile, and that the screw, besides the reactions of the nut, receives the action of three forces: one P directed along the axis, supposed vertical, and two forces $\frac{1}{2}Q$ at distance q from the axis forming a couple whose plane is perpendicular to this axis. The relation between Q, P, q and the dimensions of the screw when equilibrium is on the point of being broken by the moving force Q, tending to raise the screw, is sought.

We will suppose the helical surfaces in contact reduced to a very small width, and we will denote by r their mean distance from the axis of the screw.

Let $\sum N$ be the sum of the normal components of the reactions exerted on the screw by the helical surface of the nut; $f\sum N$ will be the sum of their tangential components, in the opposite direction to the motion which is on the point of beginning.

Let i be the angle that the tangent of the helix makes with the horizon. To express the equilibrium of translation parallel to the axis, we will have

$$\sum N\cos i - f\sum N\sin i - P = 0$$

and, taking this axis as the axis of moments, we will also have for equilibrium

$$Q\,q - r\sum N\sin i - r\,f\sum N\cos i = 0$$

from which we obtain . . . . . $\sum N = \dfrac{P}{\cos i - f\sin i}$

and consequently . . . . $Q\,q\,P\,r\,\dfrac{\sin i + f\cos i}{\cos i - f\sin i} = P\,r\,\dfrac{\tan i + f}{1 + f\tan i}$ . . . . . [1]

Example: $f = 0.12$; $\tan i = \frac{1}{25}$;

$$Q\,q = P\,r\,\frac{0{,}04 + 0{,}12}{1 - 0{,}0048} = 0.1608\,P\,r\ ;$$

while if friction were zero, we would have $Q\,q = 0{,}04\,P\,r$.

202. Calling h the pitch of the screw, we have $\tan i = \frac{h}{2\,\Pi\,r}$ and consequently

$$Q\,q = P\,r\,\frac{h + 2\,\Pi\,r\,f}{2\,\Pi\,r - fh}$$ . . . . . [2]

If friction could be neglected, we would have, as indicated in treatises on statics and as is seen immediately by the theorem of virtual motions, $Q = \frac{P\,h}{2\,\Pi\,q}$, a quantity independent of r;

but for this it would be necessary that in [1], f be very small relative to *tan i*, which never happens.

It can even happen that Q becomes larger than any assignable quantity; this would occur if $\tan i = \frac{1}{f}$; equation [1] then gives $Q = \infty$.

203. If it were a question of determining the force P capable of making the screw descend despite the resistance of the forces $\frac{1}{2} Q$, it would suffice, in formula [1], to change the sign of f and solve for $P r$; we would find

$$P\, r = Q\, q\, \frac{1 + f\ tang\ i}{tang\ i - f}$$

from which it follows that the screw will only obey the force P provided tan $i > f$. On this property is based the use of pressure screws.

204. It is also from the existence of friction that the effectiveness of the lock nut results, which ensures the invariability of the assembly of a screw and its main nut; and which plays a role analogous to that of the cone in certain tenon and mortise joints.

205. In the case where *tan i* is less than *f*, one may ask what must be the value of each force $\frac{1}{2} Q$ acting in the direction of descent for the screw to be on the point of descending; it will suffice in this case to change at the same time the sign of Q and that of *f*, in equation [1], and we will have

$$Q\, q = P\, r\, .\, \frac{f - \tan i}{1 + f \tan i}\, .$$

206. The preceding equations hold in the case where the screw moves with uniform motion.

During one complete revolution, the work of Q is $2\, \Pi\, q\, Q$, and the work of *P,* taken absolutely, is $P\, h$. We therefore have

$$\frac{\tau\{Q\}}{\tau\{P\}} = \frac{Q\, .\, 2\, \Pi\, q}{p\, h} = \frac{Q\, q}{P\, r} \qquad \frac{2\, \Pi\, r}{h} = \frac{Q\, q}{P\, r} \qquad \frac{1}{\tan i}\, .$$

or by virtue of equation [1]

$$\frac{\tau\{Q\}}{\tau\{P\}} = \frac{\tan i + f}{\tan i - f\, \tan^2 i} = 1 + \frac{f\left(1 + \tan^2 i\right)}{\tan i - f\, \tan^2 i} \ldots\ldots [\,3\,]$$

This is what would be obtained using only equation [1] by noting that the ratio of the works of Q and P is equal to the ratio of these forces multiplied by a constant number determined by the condition that the work ratio becomes unity when $f = 0$.

The work consumed by friction is therefore to the work of *P,* i.e., to the useful effect, in the ratio expressed by

$$\frac{f\left(1 + \tan^2 i\right)}{\tan i - f\, \tan^2 i}\, .$$

tan i is ordinarily a small number, and then formula [3] reduces approximately to

$$\frac{\tau\{Q\}}{\tau\{P\}} = 1 + \frac{f}{\tan i} .$$

Example: $f = 0.12$; $\tan i = \frac{1}{25}$;

$$\tau\{Q\} = \tau\{P\}\left(1 + 0{,}12\ \frac{626}{24.88}\right) = 4.02\ \tau\{P\} ;$$

or approximately . . . . . $\tau\{Q\} = 4\ \tau\{P\}$.

207. The ratio of the motive work of Q to the resistant work of P could be diminished by increasing *tan i*.

For example, leaving $f = 0.12$ and taking $tan\ i = 0.9$, we would have

$$\frac{\tau\{Q\}}{\tau\{P\}} = 1 + \frac{0.12 \times 1.81}{0.9 - 0.12 \times 0.81} = 1.27$$

The loss, due to friction, would be, as can be seen, not very considerable, since it would be 27 hundredths of the useful work; but the moment of the force Q would no longer be very small compared to $P\ r$, and thus the advantage sought in the screw would be lost; for we would have

$$Q\ q = P\ r\ .\ 1.27 \times 0.9 = 1.143\ P\ r\ .$$

208. Throughout this paragraph, only the case where the screw is acted upon by a force along the axis, and by a couple perpendicular to this axis, has been considered. In any other case, there would be lateral frictions to consider, either against the nut, or against other guides of the screw.

§: 7. Friction of crank pins or eccentrics against their bearings or bushings.

209. A crank shaft can be considered as a windlass, with this sole difference that some of the forces acting on it act through connecting rods. The question of equilibrium of such a system, considering friction, would present no theoretical difficulty. It would only be a matter of introducing into the calculations the reciprocal actions of the surfaces in contact, remembering that at the instant when motion is on the point of beginning, the forces received by one of the bodies from the one touching it decompose into normal forces N and tangential forces $f\ N$, the direction of the latter being opposite to that of the motion of the first body relative to the second. The equilibrium conditions of each piece of the system would be written, then the unknown pressures eliminated.

210. We will confine ourselves to considering the friction of a crank in the state of motion, from the point of view of the resistant work that this friction develops.

In general, when two bodies are in contact and both move, exerting on each other molecular actions, the total work resulting from these actions depends only on their intensities and on the relative motion of the two bodies; from which it follows that if one of them is reduced to rest by modifying the absolute motion of the other so that the relative motion is not changed,

and that the reciprocal actions remain the same, the work due to these actions will not be altered.

211. Let O (fig. 60) be the fixed axis of the shaft. A is the center of the movable pin, describing a circle or an arc of a circle about O. Let $A_0, A_1, A_2, \ldots\ldots$ be successive positions of the pin axis; $A_0B_0, A_1B_1, A_2B_2, \ldots\ldots$ the corresponding directions of the connecting rod; $P_0, P_1, P_2, \ldots\ldots$ the forces the connecting rod then exerts on the pin. Transfer the radius OA_0 to oa, (fig. 61) then through point a draw $ab_0, ab_1, ab_2, \ldots\ldots$ making angles b_0ao, b_1ao, $b_2ao \ldots$ equal to the angles $B_0A_0O, B_1A_1O, B_2A_2O \ldots\ldots$ and draw around a a circle representing that of the crank pin. If we suppose that a connecting rod takes successively the directions $ab_0, ab_1, ab_2, \ldots\ldots$ exerting the forces P_0, P_1, P_2 on a ring embracing the circle, the work of friction will be precisely that which is to be calculated.

Now the intensity of friction at each position is $f\,P$; the path described by the point of application of this resistant force is composed of the elements of the arc $b_0, b_1, b_2, \ldots\ldots$ from which we conclude that the work is obtained by plotting on an abscissa axis the lengths b_0b_1, b_1b_2, (fig. 62) then the forces $f_1\,P_0, f_1\,P_1, f_1\,P_2, \ldots\ldots$ as ordinates and finally calculating the area of the curve thus constructed.

212. In the particular case where the connecting rod would remain parallel to itself and exert a constant force P, the work of friction for one complete revolution of the crank would evidently be $2\,\Pi\,r\,.\,f_1\,P$.

It is therefore important not to increase the radius r of the pin beyond necessity; hence eccentrics should not be used to exert large forces.

§: 8. Mixed friction of rolling and sliding; gears.

213. Let MN (fig. 63) be a cylindrical surface forming the boundary of a body supposed fixed; let $ABCD$ be the cylindrical surface bounding another body which moves while always touching the first. If, in any second position, the movable body came to $A'B'C'D'$ (the points A, B, C, D being respectively transported to A', B', C', D') in such a way that the arc $B'C'$ of the curve MN had the same length as the arc BC of the curve $ABCD$, there would be simple rolling; and the bodies in contact, being supposed hard and polished, friction could ordinarily be neglected. But if the second position is A'', B'', C'', D'', such that BC'' is greater than BC there would be both rolling and sliding, and the difference between the arcs BC'' and BC is what is called the sliding arc.

In this case, it is admitted, although this opinion is not supported by special experiments, that the resistant or negative work due to the mutual actions of the bodies in contact is the same as if the movable body had first been transported by simple rolling from $ABCD$ to $A'B'C'D'$ and then by simple sliding from $A'B'C'D'$ to $A''B''C''D''$. It follows from this that if rolling friction is neglected, calling R the reaction of one of the bodies on the other, N its normal component, $f\,N$, or what amounts to the same, $\frac{f\,R}{\sqrt{1+f^2}}$ or $f_1\,R$, will be its tangential

component; and that the negative work due to friction in the passage from $ABCD$ to $A''B''C''D''$ will be $f_1 R \times C'C''$.

It will be noted that if the displacement of the movable body is infinitely small, the points C, B'' can be considered as situated on MN, and we have

$$C'C'' = BC'' - BC' = BC'' - BC = BC'' - B''C'' = BB''$$

Thus the sliding arc is the elementary path traveled by point B, considered as belonging to the movable body, during a very short time, from the instant this point was in contact.

214. Simultaneous rolling and sliding, instead of being in the same direction, as in the preceding example, can be in opposite directions. $ABCD$ (fig. 64) being still the first position of the movable body, and $A'B'C'D'$ the position it would take if it had simply rolled on MN, the second position can be $A''B''C''D''$, the same letters with different accents always representing the material points of the movable body. It could even happen that while rolling tends to carry the point of contact to the right of point B, sliding carries it to the left.

Whatever the case may be, it is still admitted that the negative work due to friction is expressed, as in the preceding case, by $f_1 R \times C'C''$, i.e., it is the same as if the body, having started from position $A'B'C'D'$, had been transported by simple sliding to $A''B''C''D''$; and if the displacement is infinitely small, the distance from B'' to the curve MN being an infinitesimal of the second order, we have $C'C'' = B'B''$.

215. The hypothesis just considered, different at first glance from the first, is nevertheless contained within it; for, if we suppose that the body $ABCD$ is fixed and that MN, having become movable, takes the same relative positions as before, we will see that, within the framework of No. 213, the rolling and sliding of MN would be in opposite directions; while they would be in the same direction in the case of No. 214.

216. *Application to gears* — Wheel AB (fig. 65) drives wheel $A'B'$. Contact occurs at T on the line of centers, and passes to T' while the wheels advance by one pitch. If R is the intensity of the mutual action of the surfaces in contact, and a the length of the pitch TM or TM', the work transmitted to the driven wheel differs very little from Ra, i.e., from what it would be if the force R acted tangentially to the pitch circle.

The negative work of friction, depending on relative motion, is equal to $f_1 R\ (MT' - TM')$ or $f_1 R \times MM'$, point M' being considered as situated on the curve MT. Now the arcs TM and TM' being equal, and moreover supposed small compared to their radii, the ordinates LM, LM' differ very little from each other, and the distance MM' can be replaced by LL' or $TL + TL'$.

But, if we call r and r' the radii of the two pitch circles, which touch at T, and consider the chords TM, TM' as coinciding with the arcs they subtend, i.e., as equal to the pitch a, we obtain

$$TL = \frac{a^2}{2r}, \quad TL' = \frac{a^2}{2r'},$$

and the work of friction becomes

$$\frac{1}{2} f_1 R a^2 \left(\frac{1}{r} + \frac{1}{r'} \right);$$

thus the motive work necessary to transmit to the driven wheel the work R a, and simultaneously overcome the friction of the gear, is

$$R a + \frac{1}{2} f_1 R a^2 \left(\frac{1}{r} + \frac{1}{r'} \right)$$

or

$$R a \left[1 + \frac{1}{2} f_1 a \left(\frac{1}{r} + \frac{1}{r'} \right) \right];$$

or else, calling τ the motive work and τ' the work transmitted to the second wheel

$$\tau = \tau' \left(1 + \frac{1}{2} f_1 a \left(\frac{1}{r} + \frac{1}{r'} \right) \right).$$

By these formulas, due to Mr. Poncelet, and of which we have only simplified the demonstration by starting from the hypothesis of a very small pitch, we see that the work consumed by friction varies, for given radii, proportionally to the pitch.

217. If the gear is cylindrical, we transform this formula by introducing the numbers n and n' of teeth of the two wheels.

We have $a = \frac{2 \Pi r}{n}$; $a = \frac{2 \Pi r'}{n}$, hence $\frac{a}{2 r} = \frac{\Pi}{n}$; $\frac{a}{2 r'} = \frac{\Pi}{n'}$

from that $$\tau = \tau' \left[1 + f_1 \Pi \left(\frac{1}{n} + \frac{1}{n'} \right) \right]$$

Example: $f\ 1 = 0.10$; $n = 20$; $n' = 40$; $f_1 \Pi \left(\frac{1}{n} + \frac{1}{n'} \right) = 0.024$

218. If the gear is bevel (fig. 66), it is necessary, following Mr. Poncelet's remark, to retain the formula of No. 216, taking for r and r' not the radii CT and CT', but the distances OT, $O'T$ measured along a perpendicular to the generatrix of contact AT, because the tooth curves do indeed touch for a short duration as if they remained in the plane OO' perpendicular to AT.

219. The above formulas suppose that the teeth only touch starting from the line of centers, and that there is never more than one tooth in contact on each wheel. If, while retaining this latter hypothesis, it were admitted that contact began half a pitch before the line of centers and ended half a pitch after, the work due to friction would be reduced by half, as can be seen by replacing a by $\frac{1}{2} a$ and then doubling the result.

But it is better, in practice, not to count on this reduction. The custom is to have contact begin one pitch before the line of centers and end one pitch after; in this case, there are always two teeth of each wheel in contact, and the formulas established above hold.

Moreover, the preceding theory does not take into account the inaccuracy of wheel division, and the inevitable deformation of teeth, from which result small shocks, and work losses which

must be added to those of friction. Various experiments on this matter would be of great interest.

220. We have just considered the friction of a gear in the case where the pitch is very small, and we said that the obtained formula applies equally whether the contact of two teeth begins at a distance of one pitch from the line of centers, or whether it begins on the line of centers, provided it always ends at a distance of one pitch from this line. It would no longer be the same if the teeth had to engage and disengage at a great distance from the line of centers.

Let two wheels (fig. 67) whose teeth MT', $M'T'$ touch at T'. Let us seek the condition of equilibrium between the two forces Q, P and suppose that Q is the moving force; contact in this case occurs after the line of centers.

The wheel with center C' is in equilibrium under the action of force P, the normal reaction N' and a tangential reaction $f N'$ directed along $T'M'$ (see fig. 67); we therefore have, by the condition of moments

$$P\,r' - N'\,x' = 0$$

The wheel with center C is in equilibrium under the action of force Q, the normal reaction N equal to N', and a tangential reaction $f N$ directed along $T'C$; we therefore have

$$Q\,r - N\,x - f\,N\,y = 0$$

making $N' = N$ and eliminating this quantity, we find

$$\frac{Q}{P} = \frac{r'}{r}\,\frac{x + f\,y}{x'} \qquad \text{or} \qquad \frac{Q}{P} = 1 + f\,\frac{y}{x} \quad \ldots\ldots\ldots\ldots\ldots [\,1\,]$$

noting the relation $$\frac{r'}{r} = \frac{x'}{x}$$

Now suppose on the contrary that P is the moving force. Contact then occurs before the line of centers, and the only difference to be made from the previous case will be in the direction of the tangential reactions, one $f N'$ being directed along $T'C'$ and the other $f N$ along TM; so that the equations will be:

$$P\,r' - N'\,x' = 0 \qquad \text{and} \qquad Q\,r - N\,x + f\,N\,y = 0$$

hence $$\frac{P}{Q} = \frac{r}{r'}\,.\,\frac{x'}{x - f\,y} \qquad \text{or} \qquad \frac{P}{Q} = \frac{1}{1 - f\,\frac{y}{x}} \quad \ldots\ldots\ldots [\,2\,]$$

In both cases the ratio of the moving force to the resisting force is greater than unity. But in the second it can become much greater than in the first. By taking the extreme hypothesis $f\,\frac{y}{x} = 1$, we would have $1 + f\,\frac{y}{x} = 2$ while $\frac{1}{1 - f\,\frac{y}{x}}$ would be ∞.

From which we conclude that it is possible to design two gear teeth which, satisfying the geometric conditions of layout, would not work when one tries to make them turn in a certain direction, i.e., by making contact begin far from the line of centers, and then seeking to bring the two teeth closer to this line. There would then be what is called a jamming.

This effect does not occur when y is small compared to x, and then the two quantities $\frac{1}{1-f\frac{y}{x}}$ and $1+f\frac{y}{x}$ are sensibly equal, since their ratio is that of 1 to $1+f^{2.}\frac{y^2}{x^2}$.

§: 9. On the stiffness of ropes.

221. When, over a pulley movable about its axis C (fig. 68), passes a rope acted upon at its ends by forces P and Q, experience proves that either during uniform motion, or at the instant when motion is on the point of beginning, the moving force Q is greater than the resistance P, by an amount which exceeds the increase due to the friction of the pulley axle on its supports.

This excess is an effect of the stiffness of the rope, and is explained either by the shape the rope takes, from which it results that the lever arm of the resistance P is greater than that of the moving force Q, or by the resistant work to be overcome to bend the rope.

It results from Coulomb's experiments (*Mémoires des savants étrangers*, vol. 10) that, abstracting from the friction of the axle, the excess $Q - P$ can be represented by the formula

$$Q - P = \frac{A + BP}{D},$$

D being the diameter of the pulley, increased by that of the rope, A and B being two quantities independent of P and D, but variable with the diameter, the nature of the rope (white or tarred), its dryness or humidity, its state of wear.

The following table, calculated by Navier, gives for certain ropes experimented on by Coulomb, the values of A and B which apply when the Kilogram is taken as the unit of force and the meter as the linear unit. These ropes were formed of three strands or smaller ropes, and the strands composed of a certain number of strands called yarns (*fils de caret*).

Table for calculating the stiffness of different ropes around a pulley of one meter in diameter.

Rope description	Diameter of ropes	Weight of ropes per meter length	Constant stiffness A	Stiffness per kg of resistance B
White ropes of 30 threads	0.020 m	0.283 kg	0·222 kg	0,0097
" 15 "	0.014	0.145	0.064	0.0055
" 6 "	0.009	0.052	0.011	0.0024
Tarred ropes of 30 threads	0.024	0.333	0.350	0.0126
" 15 "	0.017	0.163	0.106	0.0061
" 6 "	0.010	0.069	0.021	0.0026

We see from this table that the quantities A and B do not vary, with the thickness of the rope, according to the same law; (A is approximately proportional to the 4th power and B to

the 2nd power of the diameter) which proves the inaccuracy of the formula given in this regard by several authors after Coulomb. The experiments reported by this great physicist are not sufficient to allow the influence of the wear of the rope to be introduced into the formula. What is certain is that, with equal diameters, a half-worn rope always has less stiffness than a rope in good condition.

222. According to Navier (*Leçons 1ère partie*, page 43) the average resistance of ropes is 5 to 6 kg per square millimeter of section; but they should not be subjected to more than half of this force.

223. From formula No. 221 we conclude that if r and r' denote the distances from the pulley axis to the axes of the two straight parts of the rope, we have

$$P\,r' = Q\,r \qquad \text{or} \qquad P\,r' = P\,r + \frac{1}{2}(A + BP)$$

since r differs very little from $\frac{1}{2}D$.

224. Condition for equilibrium or uniform motion of the pulley, taking into account axle friction. Let

r be the radius of the pulley groove, increased by that of the rope,

r' the distance from the center of figure of the pulley to the axis of the rope portion to which the resistance P is applied,

ρ the radius of the axle if the latter is fixed to the pulley; if the axle is attached to the frame, ρ is the radius of the pulley eye,

R the total oblique reaction that the pulley receives from its supports; N and $f\,N$ are its components.

Neglecting the weight of the pulley, we have

$$R = N\sqrt{1 + f^2} = \sqrt{P^2 + Q^2 + 2\,P\,Q\cos(P\,,\,Q)}$$

and

$$Q\,r = P\,r' + f\,N\,\rho\,;$$

from this and formula No. 223 we conclude

$$Q = P + \frac{1}{2\,r}(A + BP) + f_1\frac{\rho}{r}\sqrt{P^2 + Q^2 + 2\,P\,Q\cos(P\,,\,Q)}\,.$$

In the particular case where the forces P and Q are parallel, this formula becomes

$$Q = \frac{1}{r - f_1\rho}\left(\frac{1}{2}A + \left(r + \frac{1}{2}B + f_1\rho\right)P\right),$$

equation which is of the form $Q = \alpha + \beta P$.

225. Calculation of tackles or hoists, neglecting the weight of the rope, that of the pulleys, the lateral friction of the latter, and supposing the pulleys equal and the ropes sensibly parallel (fig. 69)

Let $T_1, T_2, \ldots, T_n$ be the tensions of the ropes, not including the one on which the moving force Q acts; r the radius of each pulley increased by that of the rope; ρ the radius of the pulley holes; α and β being functions, determined as above, of the known quantities r, ρ, A, B and f_1, we will have

$$T_2 = \alpha + \beta T_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \alpha \frac{\beta - 1}{\beta - 1} + \beta T_1$$

$$T_3 = \alpha + \beta T_1 = \alpha (1 + \beta) + \beta^2 T_1 \ldots\ldots\ldots = \alpha \frac{\beta^2 - 1}{\beta - 1} + \beta^2 T_1$$

$$T_4 = \alpha + \beta T_2 = \alpha (1 + \beta + \beta^2) + \beta^2 T_1 \ldots\ldots\ldots = \alpha \frac{\beta^3 - 1}{\beta - 1} + \beta^4 T_1$$

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$$T_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \alpha \frac{\beta^{n-1} - 1}{\beta - 1} + \beta^{n-1} T_1$$

$$Q = \alpha + \beta T_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \alpha \frac{\beta^n - 1}{\beta - 1} + \beta^n T_1$$

moreover, $P = T_1 + T_2 + \ldots\ldots\ldots\ldots + T_n$

Putting at the head of these equations the equality $T_1 = \alpha \frac{1-1}{\beta - 1} + 1\, T_1$ and adding the n tensions from T_1 to T_n, we obtain

$$P = \frac{\alpha}{\beta - 1} \left(\frac{\beta^n - 1}{\beta - 1} - n \right) + \frac{\beta^n - 1}{\beta - 1} T_1$$

Eliminating T_1 between this equation and that giving the value of Q, we find

$$Q = \alpha \left(\frac{n \beta^n}{\beta^n - 1} - \frac{1}{\beta - 1} \right) + \frac{(\beta - 1) \beta^n}{\beta^n - 1} P ;$$

equation in which n is the number of ropes going from one block to the other.

If one abstracted from friction and stiffness, one would have

$$Q = \frac{P}{n}$$

226. Example taken from Mr. Poncelet's lithographed course.

Two systems of four pulleys; $n = 8$;

$$D = 0.1006;\ d = 0.018, \qquad \text{hence} \qquad r = 0.0593;\ \rho = 0.0105;$$

$f = 0.155$ (copper on iron without coating) hence $f_1 = 0.153$

We find $\alpha = 1.6039$ kg; $\beta = 1.1267$; $Q = 12.18$ kg $+ 0.288\, P$. If one neglected friction and stiffness, one should have

$$Q = 0{,}125\ P$$

The paths traversed simultaneously by the points of application of forces P and Q are in the ratio of 1 to n. It follows that

$$\tau\{P\} : \tau\{Q\} :: P : n\,Q$$

We see from the preceding formula that this ratio is not constant. In the above example, we have

$$\tau\{P\} : \tau\{Q\} :: P : 8\,Q \qquad \text{i.e.} \qquad :: P : 97^{K}\,,\ 44 + 2{,}304\,P\,.$$

According to whether P were 200 or 2000 kg, this work ratio would be that of 1 to *2.79* or 1 to *2.35*.

§: 10. *Friction of a rope or belt on a fixed cylinder.*

227. Consider first a portion of a regular prism M_0, M_1, etc., (fig. 70) pressed on the middle of its faces by bodies linked together by means of threads perpendicular to the radii OM_0, OM_1, *etc*., and which pass respectively through the points M_0, M_1, *etc*. The first of these threads has a tension equal to the resistance P; the tension Q of the last is required at the instant when motion is on the point of beginning in the direction of Q.

Let (fig. 71) T be the tension of the thread passing through M, $T + \Delta T$ that of the thread passing through M'; α half of the angle MOM'; N the reaction perpendicular to the face MM', $f\,N$ the friction corresponding to it. We have, considering the components of the tensions in the direction MM' and in the direction ON:

$$f\,N = \Delta T\,.\,\cos\alpha \qquad \text{and} \qquad N = (2\,T + \Delta T)\sin\alpha$$

hence, eliminating N,

$$\Delta T = \frac{2\,f\,T\,\tan\alpha}{1 - f\,\tan\alpha}\,.$$

When α is very small, $f\,tan\,\alpha$ can be neglected compared to unity; and if we denote by ΔS the length MM', we can replace, in the numerator, $tan\,\alpha$ by $\frac{\frac{1}{2}\Delta S}{r}$; it then becomes

$$\Delta T = \frac{f\,\Delta S}{r}\,.\,T$$

To pass from the considered case to that of a rope sliding on a cylinder, it suffices to make ΔT and ΔS infinitely small and set

$$\frac{d\,T}{T} = \frac{f\,d\,s}{r}$$

from which, integrating between the limits 0 and s, which correspond to the values P and Q of the tension T, s denoting the total arc embraced by the rope

$$\log\frac{Q}{P} = \frac{f\,s}{r}\,logs\,, \qquad \text{or} \qquad \frac{Q}{P} = (2.71828)^{\frac{f\,s}{r}}$$

228. *Remark* — The ratio $\frac{Q}{P}$ depends not on the arc s but only on the ratio $\frac{s}{r}$, and on the number of degrees of the embraced arc. It follows that the formula applies to any cylinder, and in it $\frac{s}{r}$ is replaced by the circular arc which, having *radius = 1*, measures the angle between

the two extreme normals to the embraced arc. If m is the number of degrees of this angle, the value to be substituted for $\frac{s}{r}$ will be $\frac{m\,\Pi}{180}$.

229. According to the experiments of Mr. Morin, the friction of belts appears to be independent of their width; and we have

$f = 0.47$ for ordinary belts on wooden drums

$f = 0.50$ for new belts on wooden drums

$f = 0.28$ for ordinary belts on cast iron pulleys

$f = 0.38$ for wet belts on cast iron pulleys

$f = 0.50$ for hemp ropes on wooden pulleys

230. According to the same author, a leather belt can safely be subjected to a tension of 0.25 kg per square millimeter of section.

§: 11. *On the transmission of motion by means of endless ropes or belts.*

231. Two bodies rotating about axes A and A' (fig. 72) carry pulleys on which is wound an endless rope or belt, suitably tensioned. The forces Q and P being applied respectively to the two bodies, the conditions of equilibrium at the moment when motion is on the point of beginning in the direction of the moving force Q, considering the friction of the trunnions A and A', and abstracting from the stiffness of the belt, which is usually slight, are required. For greater simplicity, the forces P and Q are supposed vertical, and the belt sensibly horizontal between the two pulleys.

If the belt is sufficiently tensioned not to slip on either of the two pulleys, calling T and t the tensions of the two parts of the belt, R and R' the radii of the pulleys, ρ, ρ' the radii of the trunnions, p and q the lever arms of forces P and Q; we will have

for pulley A: $$Q\,q-(T-t)\,R-f_1\,\rho\,\sqrt{Q^2+(T+t^2)}=0 \quad \ldots\ldots [\,1\,]$$

for pulley B: $$(T-t)\,R'-P\,p-f_1\,\rho'\,\sqrt{P^2+(T+t)^2}=0 \quad \ldots\ldots [\,2\,]$$

For the belt not to slip on either pulley, the ratio $\frac{T}{2}$ must be smaller than the value calculated in No. 227 for the case where slipping is on the point of beginning. Thus we must have simultaneously

$$\frac{T}{t} < e^{\frac{f\,S}{R}} \text{ and } \frac{T}{t} < e^{\frac{f\,S'}{R'}}$$

S and S' being the arcs embraced by the belt on pulleys A and A'.

In general, the second members of these inequalities may differ for two reasons: either because the angles expressed by $\frac{S}{R}$ and $\frac{S'}{R'}$ differ, or because the material of the pulleys not being the same, the friction coefficient f does not have the same value in both cases.

In the hypothesis of the statement we have

$$\frac{S}{R} = \frac{S'}{R'} = \Pi$$

but the coefficient f may not be the same.

Let m be the smaller of the second members of the preceding inequalities; we must set

$$\frac{T}{t} < m \qquad \text{or} \qquad T = K\, m\, t \ldots\ldots\ldots [3]$$

K being less than unity and the smaller the more the apparatus is exposed to shocks. If this circumstance does not exist, we take $m = 0.9$ approximately.

Equations [1], [2], [3] provide the solution to the problem.

Equations [1] and [2] would suffice if the sum $T + t$ of the tensions were given *a priori*.

232. The sum $T + t$ of the two tensions remains constant throughout the duration of motion. This results from the fact that an elastic body of prismatic shape, subjected to a tensile force in the direction of its length, undergoes an elongation proportional to this force and to its original length, provided the force is not so considerable that the elasticity of the material is altered.

Let T' be the common tension of the two strands as it would be if the forces P and Q were not yet acting; let L be the common length of these two strands, whose molecules would then be in the same state of separation; let λ be the length each of these strands would have in their natural state, i.e., if they were not tensioned. The elongation $L - \lambda$ will correspond to the tension T', and we will have

$$L - \lambda = a\, \lambda\, T' \ldots\ldots [1]$$

denoting by a the elongation that a force of 1 kg would produce on 1 m of belt.

The tension becoming T for one of the strands, let l be the new increase in its length; we have similarly

$$L + l - \lambda = a\, \lambda\, T \ldots\ldots [2]$$

and the increase l passes onto the driving pulley.

But at the same time the part of the belt which formed the second strand does not retain, in its new state, the length L except by the addition of the quantity l which unwinds from the driving pulley. t being the tension corresponding to the elongation $L - l - \lambda$, we have

$$-l - \lambda = a\, \lambda\, t \ldots\ldots [3]$$

Now, from equations [1], [2], [3], by first adding the last two, we obtain $T + t = 2\, T'$.

Therefore the sum $T + t$ of the tensions is the same at any instant as at the moment when the pulleys were not yet acted upon by the forces P and Q; i.e., it is double the initial tension of one of the strands.

The initial tension of the belt can be adjusted, either by varying its length until it is suitably tensioned, or by a device allowing the pulley axes to be moved apart at will. In general, one should avoid tensioning the belt much more than necessary, because the resultant

of the two tensions produces a pressure on each axis, and consequently gives rise to an increase in friction.

233. Independently of the tension variations due to motion, the belt undergoes others which depend on the hygrometric state of the atmosphere. It is to remedy this that a tensioning pulley is used (fig. 73).

Let N be the force exerted by the tensioning roller, and α the obtuse angle formed by the two parts of the belt on which it acts. The force N being sensibly directed along the bisector of angle α and T being the tension of the two parts of the belt strand on which the roller acts, we have approximately

$$2\,T\cos\frac{1}{2}\,\alpha = N$$

a relation which allows T to be deduced from N, or N from T.

234. Flat belt pulleys are not cylindrical, but slightly crowned: in this way the belt is a little more tensioned in the middle than at the edges, which maintains its regular position. The crowning of two unequal pulleys in communication must be proportional to the radii so that the velocities are equal on any two circumferences situated on the two pulleys in the same plane perpendicular to the axes. The average crowning of the two pulleys is about $\frac{1}{10}$ of their width.

235. An endless belt being in motion, if the resistance P happens to increase to the point where the relation $\frac{T}{t} < m$ of No. 231 is no longer satisfied, the belt will slip. This is what happens in sawmills, when the saw encounters a knot. If the living power possessed by the saw is great enough to equal the excess work required to saw this knot, the saw will overcome the obstacle by slowing down, and will then regain its original velocity; but if the living power of the saw is insufficient, it will stop. The workman, warned by this interruption, can, by pressing on the tensioning roller, impart to the saw the increase in pressure necessary to overcome the obstacle without danger of breaking the saw.

If it is desired that the operation cannot stop or slow down, gears must be substituted for belts. In this latter arrangement, the living power acquired by the whole system serves to overcome a temporary increase in resistance, without appreciable decrease in speed.

One could achieve the same goal by increasing the tension of the endless belts, but this results in a serious drawback: that of producing sometimes considerable friction on the trunnions of the rotating shafts.

If one does not wish to increase the belt tension excessively, it is at least necessary that the machine to which it communicates motion possess, by its mass and speed, a living power great enough so that the momentary increase in resulting work has little influence on the speed of this machine. These considerations relate to the theory of flywheels set forth in the following chapter.

Chap. 5. Of the varied motion of solid bodies constrained to turn about fixed axes.

§: 1 Calculation of angular acceleration.

236. External forces F act on a solid body, whose angular velocity at a certain instant is ω and after a very short time dt takes another value $\omega + d\omega$.

The quantity $\frac{d\omega}{dt}$ is the *angular acceleration* to be determined.

Let M (fig. 71) be a point having a mass m, and situated at a distance r from the axis of rotation Oz. Its velocity at the instant considered is ωr; and its acceleration is $\frac{r\,d\omega}{dt}$. This point therefore moves as if, being free, it were subjected to the action of two forces, one φ tangential to the axis described, and equal to $m\,r\,\frac{d\omega}{dt}$, the other ψ centripetal and equal to $m\,\omega^2\,r$.

That established, conceive that, without changing anything in the actual forces, we apply to point M four new forces, namely φ and ψ as defined above, and φ and ψ equal and opposite to the first ones. Do the same for all points of the rotating body; nothing will be altered in its motion nor in the mutual actions of all its parts.

In this state, the point moves as if it were acted upon only by the forces φ and ψ; the other forces F, φ and ψ, and the molecular actions relating to point M therefore mutually balance each other. The same holds for all the material points of the solid body.

The forces F, φ and ψ relating to all these points therefore satisfy the general conditions of equilibrium(*), and in particular the equation of the moment taken with respect to the axis Oz. With respect to this axis, the moment of each of the forces ψ is zero, and there remains

$$\sum M\{F\}_z = \sum \varphi\, r \qquad \text{or} \qquad \sum M\{F\}_z = \frac{\sum d\omega}{dt}\,.\,m\,r^2$$

hence

$$\frac{d\omega}{dt} = \frac{\sum M\{F\}_z}{\sum m\,r^2}$$

since at the instant considered, the angular acceleration $\frac{d\omega}{dt}$ is common to all elements m of the solid system.

We conclude that the angular acceleration is equal to the sum of the moments of the external forces with respect to the axis of rotation divided by the moment of inertia of the system with respect to the same axis.

237. *Particular case*. It may happen that $\sum M\{F\}_z$ is zero, as when a body subjected only to the action of gravity rotates without friction about an axis passing through its center of gravity, or about any vertical axis. The body then retains without variation the initial rotational motion, since $\frac{d\omega}{dt}$ is constantly zero.

(*) This very simple observation, extended to any material system, possessing any movement, basically constitutes the general principle of D'Alembert dynamics.

But it must not be concluded that each point is in this case in dynamic equilibrium; there is on the contrary for each point a centripetal force resulting from all the forces acting on this point; and if, for a given angular velocity, the molecular actions exerted on an element of the system were insufficient to produce this resultant, there would be rupture between this element and the rest of the system.

§: 2. Pressure on the supports.

238. One can propose to determine the pressures supported at each instant by the supports of the axis of rotation, or, what amounts to the same, to find the forces that the supports exert on the rotating body.

Suppose that these supports consist of two rings A and A' (fig. 72) which embrace the axis, and a plane p which opposes translation in the direction of the axis; the forces to be calculated will reduce to five forces parallel to three coordinate axes Ox, Oy, Oz: namely X, Y, components of the resistance of ring A; X', Y' analogous forces for ring A' and Z directed along the axis. These forces must be included in the general designation of external forces F considered in No. 236; but if we restrict this notation to the other external forces, then the forces φ and ψ must satisfy the six conditions of equilibrium:

1° Translation (fig. 73) . . . .

$$\sum F_x + X + X' + \sum \varphi \sin\alpha + \sum \psi \cos\alpha = 0$$
$$\sum F_y + Y + Y' - \sum \varphi \cos\alpha + \sum \psi \sin\alpha = 0$$
$$\sum F_x + Z = 0$$

2° Moments . . . . . setting $OA = \xi$, $OA' = \xi'$ and regarding as positive the moments of forces acting in the direction of the acceleration $\frac{d\omega}{dt}$

$$\sum M\{F\}_x - \sum \varphi\, r = 0$$
$$\sum M\{F\}_x + Y\,\xi + Y'\,\xi' - \sum x\,\varphi\cos\alpha + \sum x\,\psi\sin\alpha = 0$$
$$\sum M\{F\}_y + X\,\xi + X'\,\xi' + \sum x\,\varphi\sin\alpha + \sum x\,\psi\cos\alpha = 0$$

But observing that we have (fig. 73)

$$\cos\alpha = \frac{x}{r} \quad , \quad \sin\alpha = \frac{y}{r} \quad , \quad \varphi = m\,r\,\frac{d\omega}{dt} \quad , \quad \psi = m\,\omega^2\,r$$

The six equations become:

$$\sum F_x + X + X' + \frac{d\omega}{dt}\sum m\,y + \omega^2 \sum m\,x = 0 \quad \ldots\ldots\ldots\ldots\ldots [1]$$

$$\sum F_y + Y + Y'' + \frac{d\omega}{dt}\sum m\,y + \omega^2 \sum m\,y = 0 \quad \ldots\ldots\ldots\ldots\ldots [2]$$

$$\sum F_x + Z = 0 \quad \ldots\ldots\ldots\ldots\ldots [3]$$

$$\sum M\{F\}_x - \frac{d\omega}{dt}\sum m\,r^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots [4]$$

$$\sum M\{F\}_y + Y\xi + Y'\xi' - \frac{d\omega}{dt}\sum m\,x\,z + \omega^2\sum m\,y\,x = 0 \quad \ldots\ldots [5]$$

$$\sum M\{F\}_x + X\xi + X'\xi' + \frac{d\omega}{dt}\sum m\,y\,x + \omega^2\sum m\,y\,x = 0 \quad \ldots\ldots [6]$$

Equation [4] is the one used in No. 236 to determine the angular acceleration $\frac{d\omega}{dt}$; the other five will serve to determine the five unknowns *X, Y, X′, Y′,* and *Z*.

239. If we wish to use these equations for a given instant, we can choose the plane *zOx* so that it passes through the position of the center of gravity of the system at that instant. In this case we have ; $\sum m\,y = 0$ and if we denote by a the distance of the center of gravity from the axis *Ox,* by M the total mass of the body, it follows $\sum m\,x = M\,a$. We then obtain

$$\sum F_x + X + X' + \omega^2 M\,a = 0 \quad \ldots\ldots [1 \text{ bis}]$$

$$\sum F_y + Y + Y' - \frac{d\omega}{dt} M\,a = 0 \quad \ldots\ldots [2 \text{ bis}]$$

We may note that $\omega^2 M\,a$ is the expression of the centrifugal force and $\frac{d\omega}{dt} Ma$ that of the tangential force, which would correspond to the mass M, supposed concentrated at the center of gravity, without changing anything in the rotational motion.

We see from equation [3] that the pressure Z in the direction of the axis is the same during motion as if the system were at rest. It is not the same for the sums $X + X'$ and $Y + Y'$; the first, according to equation [1 bis], depends on the current angular velocity ω, and the second, according to equation [2 bis], depends on the angular acceleration $\frac{d\omega}{dt}$.

240. 1° Application of the preceding formulas to a body rotating about a vertical axis (the *x*- axis), without the action of gravity and of one or several horizontal couples.

We have in this case $\sum F_x = 0$, $\sum F_y = 0$, $\sum F_x = P = M\,g$, .
$\sum M\{F\}_x = 0$, $\sum M\{F\}_y = 0$

Consequently equations [1 bis], [2 bis], [3], [4], [5] and [6] become

$$X + X' + \omega^2 M\,\alpha = 0$$

$$Y + Y' - \frac{d\omega}{dt} M\,\alpha = 0$$

$$M\,g + Z = 0$$

$$\sum\{F\}_x - \frac{d\omega}{dt}\sum m\,r^2 = 0$$

$$Y\xi + Y'\xi' + \omega^2\sum m\,y\,\alpha - \frac{d\omega}{dt}\sum m\,x\,\alpha = 0$$

$$X\xi + X'\xi' + \omega^2\sum m\,x\,\alpha - P\,a = 0$$

If we wanted the horizontal pressures *X, X′, Y, Y′* to be zero, it would be necessary not only that $a = 0$, i.e., that the axis of rotation pass through the center of gravity, but also that $\sum m\, y\, \alpha = 0$ and $\sum m\, x\, \alpha = 0$.

This consideration explains how a flour millstone can be horizontal in the state of rest and lose this situation when it turns, and the last two formulas indicate the means of correcting this defect by introducing into vertical holes metallic masses whose position would be determined experimentally.

2° Other application, to a body subjected only to gravity and rotating about a horizontal axis passing through its center of gravity.

We have, taking the *x-axis* horizontal, $F_x = 0$, $F_\alpha = 0$, $\sum F_y = P$, $a = 0$, $\sum M\{F\}_x = 0$, $\sum M\{F\}_y = 0$, $\sum M\{F\}_\alpha = 0$.

Consequently the above-designated equations become

$$X + X' = 0,\ P + Y + Y' = 0,\ Z = 0,\ \frac{d\omega}{dt} = 0$$

$$Y\xi + Y'\xi' + \omega^2 \sum m\, y\, \alpha = 0,\ X\xi + X'\xi' + \omega^2 \sum m\, x\, \alpha = 0$$

In these two cases, the formulas would be greatly simplified if the sums $\sum m\, y\, \alpha$, $\sum m\, x\, \alpha$, were zero;

The condition in question is approximately fulfilled in a very frequent practical case, that where the body is roughly symmetrical with respect to the plane *yOx* perpendicular to the axis of rotation. Indeed, to any material point with coordinates *x*, *y*, *α*, there then corresponds another point for which *x* and *y* are the same, and *α* equal and of opposite sign. For these two points, the products m *x α* are therefore equal and of opposite signs, as are the products m *y α*; hence the sums $\sum m\, y\, \alpha$ and $\sum m\, x\, \alpha$ are zero, being composed of elements that cancel pairwise.

Moreover, the plane *yOx* then passes entirely through the center of gravity.

In the first example we then have $Y\xi + Y'\xi' = 0$, and since we also have $Y + Y' = 0$, it follows $Y = 0$ and $Y' = 0$, and it remains to determine the pressures *X* and *X′* from the equations

$$X + X' + \omega^2 M\, \alpha = 0,\ X\xi + X'\xi' - P\, \alpha = 0$$

In the second example, we have

$$X = 0,\ X' = O,\ Y\xi + Y'\xi' = 0 \text{ and } Y + Y' + P = 0;$$

these last two equations together prove that the forces *Y* and *Y′* have their resultant equal and opposite to the weight *P*. The condition $Y\xi + Y'\xi' = 0$ is satisfied by $\xi = 0$ and $\xi' = 0$; i.e., the axis may have only a single support point, which must then be situated at the center of gravity. If ξ and ξ' are not zero, they are unequal, for if $\xi = \xi'$,the equation $\xi Y + \xi Y' = 0$ or $\xi(Y + Y') = 0$ would be incompatible with $Y + Y' + P = 0$.

241. In general, when the body is symmetrical with respect to a plane passing through the center of gravity and perpendicular to the axis of rotation, equations [5] and [6] of No. 238 simplify. The six combined equations are, in this case,

$$\sum F_x + X + X' + \omega^2 M \alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots [1]$$

$$\sum F_y + Y + Y' - \frac{d\omega}{dt} M \alpha = 0 \quad \ldots\ldots\ldots\ldots [2]$$

$$\sum F_\alpha + Z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots [3]$$

$$\sum M\{F\}_z - \frac{d\omega}{dt} \sum m r^2 = 0 \quad \ldots\ldots\ldots\ldots [4]$$

$$\sum M\{F\}_x + Y\xi + Y'\xi' = 0 \quad \ldots\ldots\ldots\ldots [5]$$

$$\sum M\{F\}_y + X\xi + X'\xi' = 0 \quad \ldots\ldots\ldots\ldots [6]$$

§: 3. On the center of percussion.

242. It is supposed that the forces F reduce to a single one, and the conditions it must satisfy for the pressures X, Y, X', Y' and Z to be zero are sought.

In this supposition, equation [3] of the previous number gives $Fz = 0$, i.e., the force F must act in a plane perpendicular to the axis of rotation.

Equations [5] and [6] give

$$M\{F\}_x = 0 \qquad \text{and} \qquad M\{F\}_y = 0$$

i.e., the force must act in the plane xOy passing through the center of gravity.

Equation [1] reduces to $F_x + \omega^2 M \alpha = 0$. It becomes simply $Fx = 0$ if the body starts from rest; i.e., in this case the force F must be perpendicular to the axis Ox; from which it follows $Fy = F$.

Equations [2] and [4] determine the situation of F. Let l be the distance from the origin to the point where the force F meets the line Ox passing through the center of gravity; equation [4] becomes $F l - \frac{d\omega}{dt} \sum m r^2 = 0$ and equation [2] reduces to $F - \frac{d\omega}{dt} M \alpha = 0$.

Eliminating $\frac{d\omega}{dt}$ between these two equations, we obtain

$$l = \frac{\sum m r^2}{m \alpha}.$$

This length, independent of the intensity of the force F, is the one found in the study of the compound pendulum for the distance from the axis of suspension to the corresponding center of oscillation. This point, through which the force F must pass so that the supports experience no pressure, bears the name of center of percussion.

243. *Example*.If the body is a straight rod, very narrow, of length λ, and the axis of rotation is situated at one of its ends, we have

$$l = \frac{2}{3}\lambda$$

Conversely, if the distance from the axis of rotation to one of the ends is one third of the length, the center of percussion is situated at the other end.

This theory explains a well-known fact: if one grips a rod tightly in the hand while this rod receives a blow, one feels a shock unless the point struck is near the center of percussion relative to an axis which would pass through the middle of the hand.

§: 4. Ballistic pendulum.

244. A body at rest, suspended from a fixed horizontal axis A (fig. 77), is supposed to be struck by another body of small dimensions, whose mass is μ and velocity v, directed in a plane perpendicular to the axis of rotation; the nature of the two bodies is such that they remain united after the impact. The angular acceleration of the system at the final instant of impact is required.

During the very short duration of the impact, the striking body will experience from the struck body a variable pressure F; and if we call u the common velocity of the two bodies at B at the final instant of impact, supposing this velocity to have the same direction as velocity v, which amounts to saying that velocity v is perpendicular to AB, we will have

$$\mu v - \mu u = \int F\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots [\,1\,]$$

this integral being taken over the short duration of the impact, as over infinitely small durations dt.

Now forces equal to the successive values of F will act on the struck body during the same infinitely small durations dt. . . . . Gravity being a negligible force, in the case which concerns us, compared to the reactions F developed by the impact, we can, during the impact, abstract from this force. Calling ω the variable angular velocity of the system during the impact and denoting by h the distance BA, we will have (236)

$$\frac{d\omega}{dt} = \frac{F\,h}{\sum m\,r^2}$$

hence

$$\omega' = \frac{h}{\sum m\,r^2}\int F\,dt \quad \ldots\ldots\ldots\ldots\ldots [\,2\,]$$

calling ω' the angular velocity at the final instant of impact, when the striking and struck bodies have the same velocity u.

From [1] and [2], noting the relation $u = \omega' h$, we conclude

$$\omega' = \frac{h}{\sum m\,r^2}\left(\mu v - \mu h\,\omega\right) \quad \text{hence} \quad \omega' = \frac{h\,\mu v}{\sum m\,r^2 + \mu h^2} \ldots . [\,3\,]$$

This can be expressed by saying that the angular velocity of the system at the end of the impact is equal to the moment of the quantity of motion of the striking body before impact,

with respect to the axis of suspension, divided by the moment of inertia of the total system after impact, with respect to the same axis.

245. If we know all the quantities entering into equation [3], except the velocity v of the striking body before impact, we can deduce it from the relation

$$v = \frac{\omega}{h\,\mu}\left(\sum m\,r^2 + \mu\,h^2\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots [\,4\,]$$

To apply this formula to the ballistic pendulum, denote by I the quantity $\sum m\,r^2 + \mu\,h^2$, or the moment of inertia of the system formed by the union of the two bodies after impact. The living power possessed by the system after impact will be $\frac{1}{2}\omega^2 + I$; and at this instant, the center of gravity of the pendulum is still very little displaced from the vertical line through the axis of suspension. But, in the first oscillation that the system will make, this center of gravity will rise by a certain height H which can be observed; and if we denote by P the weight of the pendulum and the striking body which has joined it, the work of gravity from the final instant of impact to the instant of the highest elevation of the center of gravity will be $P\,H$; and since the living power at this last instant is zero, we have

$$\frac{1}{2}\,\omega^2\,I = P\,H \qquad \text{hence} \qquad v = \frac{\omega\,I}{h\,\mu} = \frac{1}{h\,\mu}\sqrt{2\,I\,P\,H}\,.$$

246. I can be determined by an experimental procedure. If T is the duration of a small oscillation of the pendulum, we have $T = \Pi\sqrt{\frac{l}{g}}$, formula in which l is the length of the simple pendulum which oscillates like the compound pendulum under consideration; we therefore have

$$l = \frac{I}{M\,a}$$

M being the mass of the pendulum and a the distance from its center of gravity to the axis of suspension.

If we write $\frac{P}{g}$ in place of M, we get $l = \frac{I\,g}{P\,a}$ and consequently

$$T = \Pi\sqrt{\frac{I}{P\,a}} \qquad \text{hence} \quad I = \frac{P\,a\,T^2}{\Pi^2}$$

and finally

$$v = \frac{P\,T}{\Pi\,h\,r}\sqrt{h\,a\,H}\,.$$

247. For the axis of suspension to experience no shock, the direction of the blow must pass through the center of percussion; we must therefore have, supposing point B very close to the plane passing through axis A and the center of gravity,

$$\tau = \frac{\sum m\,r^2}{M'\,a}$$

M' here denoting the mass of the pendulum before its union with the striking body. This quantity τ will be determined by making the pendulum oscillate before the impact experiment. For this, T' being the duration of the small oscillations, we will have

$$T' = \Pi \sqrt{\frac{\tau}{g}} \qquad \text{hence} \qquad \tau = \frac{g\,T'^2}{\Pi^2}.$$

If we succeed in introducing the striking body at this distance τ from the axis of suspension, and in a position very close to the plane drawn through the axis and the center of gravity of the pendulum before the impact, the two times T and T' of the oscillations before and after impact will be sensibly equal.

§: 5. Mutual pressures of rotating bodies in varied motion.

248. Let two bodies rotate, one about axis A, the other about axis A'. They are acted upon, one by force P, the other by force Q, and moreover linked together by a cord BB' wound on cylinders whose axes are A and A', and radii R and R'.

If the linkage were by gearing, the tension T of the cord BB' would be replaced by the mutual pressure of the teeth in contact.

That established, it is supposed that, in the rotational motion of the two bodies, one driving the other, the positive or motive work of Q is not equal to the negative or resistant work of P; that consequently the motion is varied; and the intensity of the tension T is required.

If $\frac{d\omega}{dt}$ and $\frac{d\omega'}{dt}$ are the angular accelerations of the two bodies; $\sum m r^2$ and $\sum m' r'^2$ their moments of inertia with respect to the axes of rotation A and A'; p and q the lever arms of forces P and Q, we will have (No. 236)

$$\frac{d\omega}{dt} = \frac{T R - P p}{\sum m r^2} \qquad \text{and} \qquad \frac{d\omega'}{dt} = \frac{Q q - T R'}{\sum m' r'^2}$$

Moreover, points B and B' have the same velocity, and consequently the same acceleration; we therefore have $R\omega = R'\omega'$ or by differentiating and substituting for $d\omega$ and $d\omega'$ their values,

$$R \cdot \frac{T R - P p}{\sum m r^2} = \frac{Q q - T R'}{\sum m'} \cdot r'^2$$

an equation from which we obtain

$$T = \frac{P p R \cdot \sum m' r'^2 + Q q R' \sum m r^2}{R^2 \sum m' r'^2 + R'^2 \sum m r^2}$$

249. To simplify this expression, replace the force P by a force P_1 tangent to the cylinder whose radius is R and capable of producing the same work; replace similarly the force Q by an equivalent force Q_1 tangential to the cylinder whose radius is R'; let further M and M' be two masses which, united in two rings of radii R and R', would have the same moments of inertia as the two proposed bodies; we will have

$$P\,p = P_1\,R\ ;\ Q\,q = Q_1\,R'\ ;\ \sum m\,r^2 = M\,R^2\ ;\ \sum m\,r^2 = M'\,R'^2\ ;$$

and using these values, the expression for T becomes

$$T = \frac{M'\,P_1 + M\,Q}{M' + M}\ ^{(*)}$$

By the same substitutions, the expressions for angular accelerations become

$$\frac{d\,\omega}{d\,t} = \frac{T - P_1}{M\,R} \quad \text{and} \quad \frac{d\,\omega'}{d\,t} = \frac{Q_1 - T}{M'\,R'}$$

In the case where Q_1 is equal to P_1, we find $T = P_1 = Q_1$, and consequently $\frac{d\,\omega}{d\,t} = 0$ and $\frac{d\,\omega'}{d\,t} = 0$, i.e., the motion is uniform.

In the case where Q_1 is different from P_1, let $Q_1 = P_1 + Q_2$; it follows that

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(*) I owe to Mr. Coriolis the idea of this transformation of the tension into the pressure T, the expression of which thus becomes easy to state in ordinary language.

The same formula extends to any number of rotating bodies linked together by ropes or gears.

Let, for example, fig. 79 be four winches rotating around the axes A, B, C, D; and let T be the tension of the rope Ef which links the second winch to the third.

Considering for a very short time the system of the first two winches, as subjected to the only two external forces T and P, we have for this duration the equation of the effect of the work :

$$\mathrm{T}\{T\} - \mathrm{T}\{P\} = \frac{1}{2}\sum m\,v^2 - \frac{1}{2}\sum m\,v_0^2$$

the second member of this equation being extended to all the material points of the two winches A and B.

Let P_1 be the force that would do the same job as P if it were applied in E to the second winch while P is applied to the first.

Let M be the mass which, if it were concentrated at the same point E, or distributed over the circumference passing through this point, would have the same live power as the system of the two winches A, B.

Let σ be the path described during the time Θ by the point E and by all the points of the string $E\,F$.

Finally, let u_0 and u be the speeds of E at the two extreme instants of time.

We'll have $\mathrm{T}\{P\} = P_1\,\sigma$; $\mathrm{T}\{T\} = T\,\sigma$; $\frac{1}{2}\sum m\,v^2 - \frac{1}{2}\sum m\,v_0^2 = \frac{1}{2}M\left(u^2 - u_0^2\right)$

values, the substitution of which in the previous equation gives $\sigma\,(T - P) = \frac{1}{2}\left(u^2 - u_0^2\right)M$.

By calling in the same way Q_1 the force equivalent to Q but applied at E and M the mass which concentrated at F would have the same live power as the two winches $C\,D$, we will have

$$\sigma\,(Q_1 - T) = \frac{1}{2}\left(u^2 - u_0^2\right)M'\ .$$

now the last two equations give $\frac{Q_1 - T}{M}' = \frac{T - P}{M}$ hence $T = \frac{M'P_1 + MQ_1}{M} + M'$ the general formula that it was intended to establish. It will be easy to calculate the quantities P_1, Q_1, M, M' when the forces P, Q, their distances from the axes A, D, their moments of inertia of the winches relative to their axes are known, and finally the spokes of the gear wheels of the winches or the spokes of the cylinders to which the communication cords are tangent.

$$T = P_1 + \frac{M\,Q_2}{M' + M} = P_1 + Q_2\,\frac{1}{\frac{M'}{M} + 1}$$

and consequently the angular accelerations are positive or negative according as Q is greater or smaller than P_1, which is obvious *a priori*; but from this value of T we also conclude that if one of the forces Q or P, initially supposed in equilibrium, as they must be for uniformity of motion, happens to receive a considerable increase, ordinarily of short duration, it is important that the moment of inertia of the body to which this force is applied be as large as possible relative to that of the other body, if it is desired that the tension T experience little variation.

250. This theory is applicable to the choice of the location of flywheels on shafts: these parts possessing large moments of inertia should be placed close to the points of application of forces susceptible to considerable and short-duration increases.

§: 6. On flywheels.

251. A machine composed of solid rotating parts, linked together by gears or endless cords, can have a uniform motion for each of its material flywheels; it is necessary and sufficient for this that the sum of motive work during any time be constantly equal to the sum of resistant work during the same time.

When this equality does not hold for any equal durations, it can nevertheless exist for certain periods; then the variable velocities periodically become the same again.

If v_0 is, at a certain instant, the velocity of a mass element m of the system; and if it becomes v after any time interval, τ_m being for this same time interval the motive work of the forces acting on the machine, and τ_r the absolute value of the resistant work, we have (39)

$$\sum \frac{1}{2}\,m\,v^2 - \sum \frac{1}{2}\,m\,v_0^2 = \tau_m - \tau_r$$

equation in which the quantities $\sum \frac{1}{2}\,m\,v_2$, τ_m and τ_r are variables depending on the time elapsed since the initial instant.

If we suppose that τ_m first increases more rapidly than τ_r in the early times following the initial instant, the velocities v will increase, and the greatest value of $\sum \frac{1}{2}\,m\,v_2$ will correspond to the greatest value of $\tau_m - \tau_r$; if on the contrary it is τ_r which begins to prevail, the velocities v will decrease, and the minimum of $\sum \frac{1}{2}\,m\,v^2$ will correspond to the maximum of $\tau_r - \tau_m$, or of the absolute value of the difference between resistant work and motive work.

The instant when this difference is a maximum is also that when its increase is zero for an infinitely small time, i.e., it is the instant when the elementary motive work is equal to the elementary resistant work; in other words still, it is the instant when the forces satisfy the conditions of equilibrium.

Hence a means of establishing a relation between the greatest and smallest angular velocity of one of the rotating parts, when the forces are known for all positions of the system.

252. *First example*: Single-acting crank (fig. 80).

A wheel having its center of gravity on the axis O, is subjected to two forces: one P, resistant, acts constantly at the end of the lever arm $DO = a$; the other Q, motive, applied to the crank pin B with radius $OB = b$, acts in a constantly vertical direction, but only downwards.

For the motion to be periodically uniform, we must have, after one complete revolution

$$\tau\{P\} = \tau\{Q\} \quad \text{or} \quad P\ 2\ \Pi\ \alpha = Q\ .2\ b \ldots\ldots\ldots\ldots\ldots\ldots [\ 1\ ]$$

Starting from the instant when the pin is at A, the motive work is at first smaller than the resistant work, for the elementary works are proportional to the moments (28), and when the pin is at A, the moment of Q is zero. From this instant, the velocities decrease until the instant when the forces P and Q satisfy the conditions of equilibrium. Let x be the angle $B'OE$ that the crank radius then makes with the horizontal; we will have

$$P\ \alpha = Q\ b \cos x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [\ 2\ ]$$

These equations [1] and [2] give $\cos x = \frac{1}{\Pi}$ hence $\log \cos x = -\log \Pi = 0.4971499 = \bar{1}\ .5028501$ and $x = 71\ °\ 26\ °\ .4$

The point B'', where the moment of Q, after having exceeded that of P, becomes equal to it again, is on the same vertical as B'. Thus the velocities are at their minimum when the crank pin is at B', and at their maximum when it is at B''.

Let ω' and ω'' be the values of the angular velocity at these two instants; we will have, by virtue of the theorem of the effect of work,

$$\frac{1}{2}\omega''^2 \sum m^2 - \frac{1}{2}\omega'^2 \sum m\ r^2 = Q \times B'\ B'' - P \times arc\ M'\ E\ M''$$

now

$$B'\ B'' = 2\ b \sin x = 0.94799\ .\ 2\ b$$

$$M'\ E\ M'' = 2\ \Pi\ \alpha\ \frac{71\ °\ 26'\ .4}{180} = 2\ \Pi\ \alpha\ \frac{4286.4}{10.800} = 0.3989\ .\ 2\ \Pi\ \alpha$$

substituting, having regard to equation [1], it becomes

$$\frac{1}{2}\left(\omega''^2 - \omega'^2\right) \sum m\ r^2 = 0{,}5511\ .\ 2\ b\ Q \ldots\ldots\ldots\ldots [\ 3\ ]$$

a relation which allows calculating what the moment of inertia $\sum m\ r^2$ of the wheel must be so that the difference of the extreme velocities ω'' and ω' (which can be regarded as differing very little from the mean angular velocity of the wheel, deduced from the number of revolutions it makes per minute). Let us impose the condition that the difference $\omega'' - \omega'$ be at most $\frac{1}{R}$ of this mean; we will have

$$\frac{1}{2}\left(\omega'' - \omega'\right) = W \qquad \text{and} \qquad \omega'' - \omega' = \frac{W}{R}$$

hence $$\frac{1}{2}\left(\omega''^2 - \omega'^2\right) = \frac{W^2}{R}$$

and $$\frac{W^2}{R}\sum m\,r^2 = 0.5511\,.\,2\,b\,Q \quad \ldots\ldots\ldots\ldots [\,4\,]$$

From this relation we will deduce $\sum m\,r^2$ as a function of W, of n, and of the work $2\,b\,Q$ that the moving force develops each revolution.

253. If we suppose that the mass of the wheel can be considered as concentrated in a ring whose weight is Π, the mean velocity W, and the radius R, we will have

$$w = \frac{W}{R} \qquad \text{and} \qquad \sum m\,r^2 = \frac{\Pi}{g}R^2$$

hence $$w^2\sum m\,r^2 = \frac{\Pi\,W^2}{g}$$

Equation [4] thus gives

$$\Pi\,W^2 = 0.5511\,.\,2\,b\,Q\,.\,gn \quad \ldots\ldots\ldots\ldots [\,5\,]$$

Let M be the number of revolutions of the crank per minute, and N the number of horsepower expressing the indefinitely prolonged work of the force Q; the work per second will be expressed at the same time by $\frac{2\,b\,Q\,M}{68}$ and by $75\,N$

hence $2\,b\,Q = 4.500\,\frac{N}{M}$; and equation [5] reduces to

$$\Pi\,W^2 = 24300\,\frac{n\,N}{M} \quad \ldots\ldots\ldots\ldots\ldots [\,6\,]$$

254. *2nd Example*. Double-acting crank.

The force Q, of constant intensity and direction, acts from top to bottom in the half-circumference $AB'BA''$ and from bottom to top in the other half-circumference that the pin describes while rising.

The motion having to be periodically uniform, we must have for one complete revolution

$$\mathrm{T}\{P\} = \mathrm{T}\{Q\} \qquad \text{or} \qquad P\,.\,2\,\Pi\,\alpha = Q\,.\,4\,b \quad \ldots\ldots [\,7\,]$$

and it follows that the motion is also periodically uniform by half-revolutions; for in each of them the resistant work is $P\,\Pi\,\alpha$ and the motive work $Q\,2\,b$, equal quantities according to equation [7].

The points B' and B'', or B'' and B^{IV}, position of the pin when the velocities are at minimum and maximum, are also determined by the equation

$$P\,\alpha = Q\,b\cos x \quad \ldots\ldots\ldots\ldots\ldots [\,8\,]$$

Conditions [7] and [8] give

$$\cos x = \frac{2}{\Pi}, \qquad \text{hence} \qquad x = 50^\circ\,.\,27'\,.\,6$$

We will find, operating as in No. 252 and having regard to relation [7]

$$\frac{1}{2}\left(\omega''^2 - \omega'^2\right)\sum m\, r^2 = Q\,.\,2\,b\,\sin x - P\,2\,\Pi\,\alpha\,.\,\frac{50\,°\,27\,'\,6}{180} = Q\,2\,b\,(0.7712 - 0.5607) = 0.2105\,.\,2\,b\,Q$$

and, keeping for ω and n their meanings,

$$\frac{w^2}{n}\sum m\, r^2 = 0.2105\,.\,2\,b\,Q$$

255. Π, W, M and N having the same meanings as in No. 253,

we will have $\quad W^2\sum m\, r^2 = \dfrac{\Pi\, W^2}{g}$ $\quad$ and $\quad \dfrac{4\,b\,Q\,M}{60} = 75\,N$

and finally

$$\Pi\, W^2 = 4645\,\frac{n\,N}{M} \quad \ldots\ldots\ldots\ldots\ldots\ldots [\,9\,]$$

256. Application to a steam engine cited by Mr. Morin.

Flywheel radius $\quad R = 3.05\ m;\ M = 19$, $\quad$ hence $\quad W = \frac{2\,\Pi\,R\,.\,19}{60} = 6.06$.

$$N = 35, \qquad n = 40, \quad \text{hence} \quad \Pi = 9320\ kg.$$

257. *3rd Example*. Single-acting crank but with counterweight (fig. 81)

At a point K diametrically opposite the crank pin B, and at any distance c, is fixed a body whose weight C is such that $Cc = \frac{1}{2}\,Q\,b$.

It results that, during the descent of the pin, half of the work of Q is destroyed by that of C; but in the following half-revolution, the force Q no longer acting and body K descending, its weight C restores half of the work of Q, which it had neutralized during the first half-revolution.

The motion is therefore the same as in the case of the simple double-acting crank, and consequently the formulas of Nos. 254 and 255 are those which it is appropriate to apply.

It will be observed that if Π does not include the weight C, we must, calling v the mean velocity of this additional body, or of this weight, set the equation

$$\Pi\, W^2 + C\, v^2 = 4645\,\frac{n\,N}{M} \quad \ldots\ldots\ldots\ldots [\,10\,]$$

258. *Remark*. Formulas [6], [9] and [10] are applicable to the case where the flywheel is not on the crank shaft itself, provided M is always the number of revolutions made by the crank per minute, while W and v would be the proper velocities of the flywheel ring, and of the counterweight C always diametrically opposite the crank pin.

259. *4th Example*. Double-acting crank with counterweight (fig. 82).

The usefulness of the counterweight for the single-acting crank raises the question of whether an analogous arrangement could not be advantageously applied to a double-acting crank.

Evidently, in this latter case, the counterweight could not be fixed to the crank shaft, for if it came to the aid of the moving force Q when the crank pin was at one of the points A, A' (commonly called dead points), it would become resistant when the pin was at the other of these points. But the question receives on the contrary an affirmative solution if the counterweight is placed on a shaft linked to the crank shaft by gearing so that it makes two revolutions while the crank makes one.

O being the crank axis, let O' be a parallel axis linked to the first so that it has double the angular velocity.

Let C be the weight of a body having a center of gravity at a distance from the axis O' and solidly fixed to this axis so that when the crank pin is at the highest point A, the body C is at H' on the horizontal $O'H'$ passing through O', and in the process of descending; it follows that when the pin is at H on the horizontal OH, the body C is at H'' on the extension of the horizontal $H'O'$, and in the process of rising. In general, the weight C becomes motive when the work of Q is near its minimum, and resistant in the opposite case. It therefore operates a temporary regulation of motion, which it will be easy to complete with a flywheel.

First, determine point C, so that the algebraic sum of the work of Q and of C, during the same very small displacement, is the same when the crank pin is in the two positions A, H, and consequently when the body C is in the two positions H', H''. This condition (noting that the angular displacement of the radius $O'C$ is always double that of OB) leads to the relation

$$Cc = \frac{Q\,q}{2} - Cc \qquad \text{or} \qquad Cc = \frac{1}{4}\,Q\,b \quad \ldots\ldots\ldots [\,1\,]$$

Let us now seek the two positions of greatest and smallest angular velocity. For a half-turn of the crank, body C makes one complete turn; the work of its weight is zero and the work of Q is as before, because of the periodic uniformity of the motion,

$$Q\,.\,2\,b = \frac{1}{2}\,\tau \quad \ldots\ldots\ldots\ldots\ldots [\,2\,]$$

calling τ the resistant work developed during one crank revolution by a constant force P that the machine is intended to overcome, and which we will suppose applied to a wheel fixed on shaft O'.

Let B be a situation of the pin where the forces Q, C and P are in dynamic equilibrium. Call x the angle AOB and consequently $2x$ the angle $H'O'C$; ω the angular velocity of the crank at this instant, and consequently 2ω the angular velocity of C. Write that the work of Q plus that of C during an infinitely small time θ is equal to that of P during the same time. We will have

$$Q \sin x \cos \theta + C \cos 2x\,.\,\cos \omega c\,\theta = \tau\,.\,\frac{\omega\,\theta}{2\,\Pi}$$

or by virtue of equations [1] and [2]

$$\sin x + \frac{1}{2}\cos 2x = \frac{2}{\Pi}$$

what changes by $$\sin x + \frac{1}{2} - \sin^2 x = 0.636$$

from which we derive two values of sin x; namely 0.163 and 0.837 and consequently four values of the angle x smaller than 180°; namely

$$x' = 9°\,23' \qquad x''' = 123°\,8'$$
$$x'' = 56°\,52' \qquad x^{IV} = 170°\,37'$$

That being so, proceeding as before (252 and 254), we will have

$$\frac{1}{2}\left(\omega''^2 - \omega'^2\right)\sum m r^2 = Q\, b\left(\cos x' - \cos x''\right) + C\, c\left(\sin 2x'' - \sin 2x'\right) - \tau\frac{x'' - x'}{360°}$$
$$\frac{1}{2}\left(\omega'''^2 - \omega''^2\right)\sum m r^2 = Q\, b\left(\cos x'' - \cos x'''\right) + C\, c\left(\sin 2x''' - \sin 2x''\right) - \tau\frac{x''' - x''}{360°}$$

and substituting the above obtained values of x, of $C\,c$, and of τ, [1] and [2], we will find

$$\frac{1}{2}\left(\omega''^2 - \omega'^2\right)\sum m r^2 = \quad 0{,}035\ .\ 2\,b\,Q\ ,$$
$$\frac{1}{2}\left(\omega'''^2 - \omega''^2\right)\sum m r^2 = -\,0{,}050\ .\ 2\,b\,Q\ ,$$
$$\frac{1}{2}\left(\omega^{IV\,2} - \omega'''^2\right)\sum m r^2 = \quad 0{,}036\ .\ 2\,b\,Q\ ;$$

hence $$\frac{1}{2}\left(\omega'^2 - \omega^{IV\,2}\right)\sum m r^2 = -\,0{,}020\ .\ 2\,b\,Q\ .$$

The order of magnitude of angular velocities is therefore: ω''', ω', ω^{IV}, ω''.

The greatest variation occurs between the second and third positions of dynamic equilibrium; and setting in the equation

$$\frac{1}{2}\left(\omega''^2 - \omega'''^2\right)\sum m r^2 = 0{,}050\ .\ 2\,b\,Q$$

the same substitutions as in Nos. 252 and 255, namely:

$$\frac{1}{2}\left(\omega''^2 - \omega'''^2\right) = \frac{W^2}{n}\ , \qquad W^2\sum m r^2 = \frac{\Pi\, W^2}{g} \quad \text{et} \quad 2\,b\,Q = \frac{2250\,N}{M}$$

we find $$\Pi\, W^2 = 1104\,\frac{n\,N}{M}\ .$$

That is to say, a quarter of what formula [9], No. 255, indicates for the double-acting crank.

260. Questions analogous to the preceding can be solved in more complicated cases, using graphical methods.

Suppose, for example, that it concerns a steam engine with a beam linked by a connecting rod to the crank of a flywheel.(*) Let us admit 1° that for each position of the piston we know the resultant F of the pressures and frictions it undergoes. 2° that the resultant of the weights of the connecting rods and the beam passes through the axis of rotation of the latter, at least in their average position, so that the work of gravity on the system in motion is approximately zero. 3° that the motion is periodically uniform, so that after each turn of the flywheel and crank, all velocities become the same again.

That established, let us take as the beginning of a period the instant when the piston is at one end of its stroke. Let at this instant ω_0 be the angular velocity of the crank shaft.

Calling b the radius OB of the crank (fig. 83), we can represent the living power of the system at this instant by

$$\frac{1}{2}\,\omega_0^2\, b^2\, M$$

denoting by M a mass fictitiously united on the circle whose radius is OB; similarly in the cases where the crank shaft is linked by gears to one or several bodies rotating about axes passing through their respective centers of gravity.

Let, at another instant, ω be the angular velocity of the crank shaft, so that the living power of the rotating parts then becomes

$$\frac{1}{2}\,\omega^2\, b^2\, M$$

Add to this the living power possessed at this instant by the beam and its connecting rods. For this purpose, call ω' the angular velocity of the beam and admit, which differs little from the truth, that its living power of the connecting rods, including the piston, is the same as if their masses were transported to the articulation points of these pieces with the beam. Hence the variable living power of the beam and connecting rods will be, at the instant considered, represented by

$$\frac{1}{2}\,\omega'^2\, c^2\, M'$$

calling c the radius AC joining the axis A of rotation of the beam to the articulation point C; and denoting by M' a mass fictitiously transported to the distance AC from this axis.

If, moreover, we call τ_m and τ_r the motive work on the piston and the resistant work on the rotating parts since the initial instant, we will have

$$\frac{1}{2}\left(\omega^2 - \omega_0^2\right) b^2\, M + \frac{1}{2}\,\omega'^2\, C^2\, M' = \tau_m - \tau_r \quad \ldots\ldots\ldots\ldots [\,1\,]$$

To calculate τ_m approximately, we can admit that the piston connecting rod DE keeps sensibly the same direction, so that the work of the force F is the same as if this force acted

(*) Mr. Coriolis has treated this question more completely in a memoir inserted in the journal of the Polytechnic School.

tangentially to a circumference having its center at A. Let r be the radius AD' of this circle; during an infinitely small time dt, the work of F will be

$$F\ \omega'\ r\ dt \quad \ldots\ldots\ldots\ldots\ldots\ldots [\ 2\ ]$$

now there is a simple relation between the velocities ω and ω': indeed, the length BC being constant, the velocities of points B and C projected onto BC are equal: therefore if we drop onto BC the perpendiculars AM, ON, we have

$$\omega'\ C \cos \widehat{CAM} = \omega\ b \cos \widehat{BON} \qquad \text{or} \qquad \omega'\ AM = \omega\ .\ ON$$

More simply, if we draw OQ parallel to AC and denote OQ by q, we have

$$\frac{AM}{ON} = \frac{AC}{OQ} = \frac{c}{q} \qquad \text{and} \qquad \omega' = \omega\ \frac{q}{c} \quad \ldots\ldots\ldots\ldots [\ 3\ ]$$

so that the ratio of ω to ω' varies proportionally to the single variable q.

It follows that formula [2] can be replaced by

$$F\ \omega\ \frac{q\ r}{c}\ dt\ , \qquad \text{or} \qquad F\ \frac{q\ r}{c\ b}\ \omega\ b\ dt\ , \qquad \text{or} \qquad \frac{q\ r}{c\ b}\ F\ dx$$

denoting by dx the arc described by point B during which the element of work τ_m occurs. Therefore finally

$$\tau_m = \int_0^x \frac{q\ r}{c\ b}\ F\ dx \quad \ldots\ldots\ldots\ldots\ldots\ldots [\ 4\ ]$$

by counting the arc x from the dead point occupied by B at the initial instant. This integral can be represented and calculated by the area of a curve, taking as abscissas the arcs x and as ordinates the values of $\frac{r}{c\ b}\ .\ F\ q$, by virtue of the variables F and q depending on x.

Substituting formula [3] into [1] gives

$$\frac{1}{2}\left(\omega^2 - \omega_0^2\right) b^2\ M + \frac{1}{2}\ \omega^2\ q^2\ M' = \tau_m - \tau_r \quad \ldots\ldots\ldots [\ 5\ ]$$

In this equation, τ_m is known for various values of x; the same holds for τ_r, which is ordinarily proportional to x. We can now, as before, set

$$\frac{1}{2}\left(\omega^2 - \omega_0^2\right) \ <\ \frac{1}{n}\ W^2\ .$$

Finally, we will have a good approximation of the term $\frac{1}{2}\ \omega^2\ q^2\ M'$ by replacing in it ω by the angular velocity W, so as to leave only the variable q depending on x in it. By this means, equation [5] gives

$$\frac{\tau_m - \tau_r - \frac{1}{2}\ \omega^2\ q^2\ M'}{b^2\ M}\ <\ \frac{1}{n}\ W^2 \qquad \text{or} \qquad b^2\ M\ >\ n\ \frac{\left(\tau_m - \tau_r - \frac{1}{2}\ W^2\ q^2\ M'\right)}{b^2\ W^2}$$

By calculating the greatest value of the second member, we will obtain the moment of inertia of the flywheel $M\ b^2$.

§: 7. On hammers driven by cams[*]

261. "In forges and other metallurgical works, three kinds of hammers are used, whose shape and proportions depend on the use for which they are intended, namely:

"1° Tilt hammers, called trip hammers, used for drawing out and plating small iron, refining steel, manufacturing bladed weapons and various tools; the weight of the hammer proper varies from 80 to 40 kg and below."

It strikes "from 200 to 400 blows per minute. The cam acts on the end of the helve opposite to that where the hammer is fixed, and the axis of rotation is between these two points".

"2° Lifting hammers, whose cam lifts the helve between the hammer and the axis of rotation. They are also called large hammers, German-style hammers. They are particularly used in refining iron by the German method. Their weight varies from 300 to 400 kg."

They strike from 70 to 200 blows per minute.

"3° Frontal hammers, so named because the cam seizes the end of the helve beyond the hammer, relative to the axis, and in the vertical plane of this helve. They are used in forges where iron is manufactured by the English method, and weigh 2500 to 4000 kg, including the helve which is made of cast iron."

They strike from 60 to 100 blows per minute.

"To this class also belong most of the hammers or stamps used for felting wool or preparing rags for papermaking."

262. "The theory of the motion of these three kinds of hammers is the same, except for some modifications to be made to the formulas concerning the position of the cam and the helve. . . . . It ordinarily happens in all cases that the impact of the cam against the helve takes place in a horizontal plane passing through the axis of rotation. We will reason on this supposition, and we will distinguish three periods in the action of the cams: the first corresponding to the duration of the impact and the reciprocal compression of the cam and the part of the helve on which it acts; the second, which begins at the instant when the impact ends and lasts until that when the cam leaves the helve; the 3rd, which elapses between the moment when the cam has left the helve and that when another one strikes it. . . . . ."

263. "From the instant when the cam meets the helve, there develop at the points of contact efforts of mutual compression which increase up to the moment when the impression reaches its maximum. At this limit, the points of contact move with the same velocity; beyond, the reaction of molecular springs tends to more or less erase the impression and to increase the velocity transmitted to the hammer by diminishing that of the cam. . . . . But on the one hand, the repetition and intensity of impacts alter the elasticity of the point in contact, and on the other, the general dimensions of the helve and the cam are such that they actually flex very

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(*) The subject of this paragraph is treated with more developments and greater accuracy in the course of mechanics of Mr. Poncelet at the school of artillery and engineering; the passages indicated by quotation marks are verbatim extracted from the lithographed sheets of this course, edition of 1836.

little; from which it follows that at the end of the greatest compression, the helve and the cam move sensibly with the same velocity. This is confirmed by observation, for in no practical case is it possible to observe the least separation between these bodies."

264. Let N be the variable pressure that the hammer helve receives from the cam at any instant of the impact period.

Let at this same instant, $\frac{d\omega}{dt}$ be the angular acceleration of the hammer considered as a solid body.

This angular acceleration, according to formula (236)

$$\frac{d\omega}{dt} = \frac{\sum M\{F\}}{\sum m r^2}$$

depends on the moments of the various forces acting on the hammer, and consequently, not only on the force N but also on gravity and friction on the trunnions. But the weight of the hammer, however considerable it may be, can be neglected compared to the force N occurring at the points of contact during impact. The same applies to friction, whose moment is very small compared to that of N.

Let R be the distance from the point of contact of the cam and the hammer to the axis of rotation of the latter.

The moment of N with respect to this axis is $R\,N$, and the preceding equation becomes

$$d\omega = \frac{R}{\sum m r^2} N\,dt$$

hence, integrating and calling ω the angular velocity of the hammer at the end of impact,

$$\omega = \frac{R}{\sum m r^2} \int N\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots [1]$$

During the same period, the camshaft is subjected to a force equal and opposite to N which it receives from the hammer at the point of contact of the latter with the cam. This results in a negative angular acceleration whose absolute value, which we will denote by $\frac{d\omega'}{dt}$, is equal to the moment of the force N with respect to the axis of rotation of the camshaft. Let $R'N$ be this moment; R' being the radius of the pitch circle of the cams. We have, denoting by $\sum m' r'^2$ the moment of inertia of the camshaft with respect to its axis of rotation,

$$d\omega' = \frac{R' N\,dt}{\sum m' r'^2} \quad \text{hence} \quad \omega'_0 - \omega' = \frac{R'}{\sum m' r'^2} \int N\,dt \quad \ldots [2]$$

ω'_0 and ω' being the angular velocities of the camshaft at the beginning and end of impact.

The two integrals $\int N\,dt$ which enter into equations [1] and [2] are equal because they are taken for the same values of N and the same time limits. Therefore, eliminating $\int N\,dt$, we obtain

$$\frac{\sum m r^2}{R} \cdot \omega = \frac{\sum m' r'^2}{R'} \left(\omega'_0 - \omega'\right) \ldots\ldots\ldots\ldots [3]$$

265. Let u be the velocity of the cam at the beginning of impact and v its velocity at the end of the same period; v is also the velocity of the contact point of the helve at the end of impact. We therefore have

$$u = \omega'_0 R', \qquad v = \omega' R' = \omega R$$

Let us further set, for simplicity,

$$\sum m r^2 = M R^2 \qquad \text{for the hammer,}$$

$$\sum m' r'^2 = M' R'^2 \quad \text{for the camshaft;}$$

so that M and M' are two masses which, substituted for those of the hammer and the camshaft and concentrated at distances R and R' from their respective axes of rotation, would have the same moments of inertia relative to these axes. Equation [3] thus becomes

$$M v = M'(u - v) \qquad \text{or} \qquad (M + M')v = M' u \ldots\ldots\ldots [4]$$

exactly as if a body having mass M' and velocity u directly struck a body at rest having mass M, and v were the common velocity at the end of impact.

266. Knowing the fictitious mass M of the hammer, we can determine M' so that the difference $u - v$ of the extreme velocities does not exceed a certain fraction of the mean velocity $\frac{u+v}{2}$.

Let $\frac{u+v}{2} = V$ and $u - v = \frac{V}{n}$; we deduce $v = V\left(1 - \frac{1}{2n}\right)$ and equation [4] gives:

$$M\left(1 - \frac{1}{2n}\right) = M' \cdot \frac{1}{n} \qquad \text{hense} \qquad M' = \left(n - \frac{1}{2}\right) M \ldots\ldots\ldots [5]$$

According to Mr. Poncelet, it is very rare in practice for M' not to be above $10M$. For trip hammers and lifting hammers used in iron manufacturing, we have at least $M' = 12M$; and for frontal hammers, at least $M' = 30M$.

267. Let us calculate the living power which has been lost by the camshaft during the very short duration of impact and which must be restored to it by the motor, during the following two periods.

At the beginning of impact, this living power is $\frac{1}{2}\omega'^2_0 \sum m' r'^2$, or $\frac{1}{2} M' u^2$ at the end, it is $\frac{1}{2} M' v^2$; the decrease, and consequently the corresponding work that the motor must perform, is therefore

$$\tau_1 = \frac{1}{2} M'\left(u^2 - v^2\right) \ldots\ldots\ldots\ldots\ldots [6]$$

Neither of the velocities u, v can be observed; but it is admitted that their arithmetic mean differs very little from the mean velocity of the cams calculated from the radius R' and the number of revolutions the shaft makes in a given time.

Let therefore

$$\frac{u+v}{2} = V \ldots\ldots\ldots\ldots\ldots\ldots [7]$$

this relation combined with equation [4] allows us to calculate, as a function of V, M and M', the value of $\frac{1}{2}(u^2 - v^2) = \frac{u+v}{2} . u - v = V (u - v) = V v \frac{M}{M'}$;

then eliminating u from [4] and [7] we find

$$v = \frac{2 M' V}{2 M' + M} ,$$

and we finally obtain

$$\tau_1 = \frac{2 M M' V^2}{2 M' + M} \quad \text{or} \quad \tau_1 = M V^2 . \frac{1}{1 + \frac{M}{2 M}} \ldots\ldots\ldots [8]$$

We have seen above that the ratio $\frac{M}{M'}$ is less than $\frac{1}{10}$ and sometimes less than $\frac{1}{30}$; thus the work τ_1 differs little from MV^2 , i.e., from twice the living power the hammer would have if its velocity were calculated from the mean velocity of the cams. Half of this work is destroyed by the resistant work of molecular actions during impact.

268. Let us now calculate the work necessary for raising the hammers.

Call

P, the weight of the hammer, its helve and its iron fittings;

h, the height through which the center of gravity of this assembly rises up to the instant when the cam leaves the hammer (after which the latter continues to rise by an amount not included in h);

y, the sensibly vertical path traversed, up to the same instant, by the point of contact of the cam and the hammer;

N', the force exerted at this point.

The velocity of the hammer varying very little during the second period, the forces P, N' and the friction on the hurasse trunnions can be considered as balancing each other.

The forces P and N' being sensibly parallel, the pressure on the trunnions is:

in tilt hammers . . . . . $P + N'$

in frontal or lifting hammers . . . . . $P - N'$

The friction will be

$$\frac{f}{\sqrt{1 + f^2}} (P \pm N') \quad \text{or} \quad f_1 (P \pm N') .$$

The work of N' is equal to that of P and of the friction: thus, calling ρ the radius of the hurasse trunnion, and noting that the path described by the circumference of this trunnion is $y . \frac{\rho}{R}$, we have

$$N' y = P h + f_1 y \frac{\rho}{R} (P \pm N')$$

hence the work exerted by the cam on the hammer

$$N' y = \frac{P\left(h + f_1 y \frac{\rho}{R}\right)}{1 \pm f_1 \frac{\rho}{R}} \quad \ldots\ldots\ldots\ldots\ldots\ldots [\,9\,]$$

269. The motor must also supply work equal to that of the friction on the camshaft trunnions and of the cam against the hammer helve.

Let P' be the pressure exerted on the camshaft trunnions, this force differing little from the weight of this shaft. If we denote by ρ' the radius of the trunnions, the work of the friction they receive during the second period is

$$f_1 P' y \frac{\rho'}{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [\,10\,]$$

According to the theory of gear friction (7), the work of cam friction during the same time is

$$N' y \,.\, f_1 \frac{y}{2}\left(\frac{1}{R} + \frac{1}{R'}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots [\,11\,]$$

In summary, the resistant work undergone by the camshaft during this period is composed of the three parts determined above [9], [10] and [11]; it is

$$\tau_2 = P' \frac{h + f_1 y \frac{\rho}{R}}{1 \pm f_1 \frac{\rho}{R}}\left(1 + \frac{1}{2} f_1 y \left(\frac{1}{R} + \frac{1}{R'_E}\right)\right) + f_1 P' y \frac{\rho}{R'} \quad \ldots\ldots [\,12\,]$$

270. The camshaft, during its idle running, is subjected to no other resisting force than that of friction on its trunnions.

Let P'' be the pressure they experience, which can be made approximately equal to the weight of the camshaft, although it must depend for a small part on the action of the motor.

If we denote by y' the path described by the cam from the instant when it has left the hammer helve until that when the following cam comes to strike it, the work of friction during this period will be

$$\tau 3 = f_1 P'' \,.\, y' \,.\, \frac{\rho'}{R'} \quad \ldots\ldots\ldots\ldots\ldots\ldots [\,13\,]$$

It will be noted that $y + y'$ forms the interval between two consecutive cams, equal to $\frac{2\,\Pi\,R'}{C}$ the number of cams being C.

271. The work to be supplied by the motor on the camshaft, whose motion must be periodically uniform, is evidently for each hammer blow equal to the sum of the quantities τ_1, τ_2, τ_3 given by formulas [8], [12] and [13].

From this, the intensity of the motive work per second will easily be deduced.

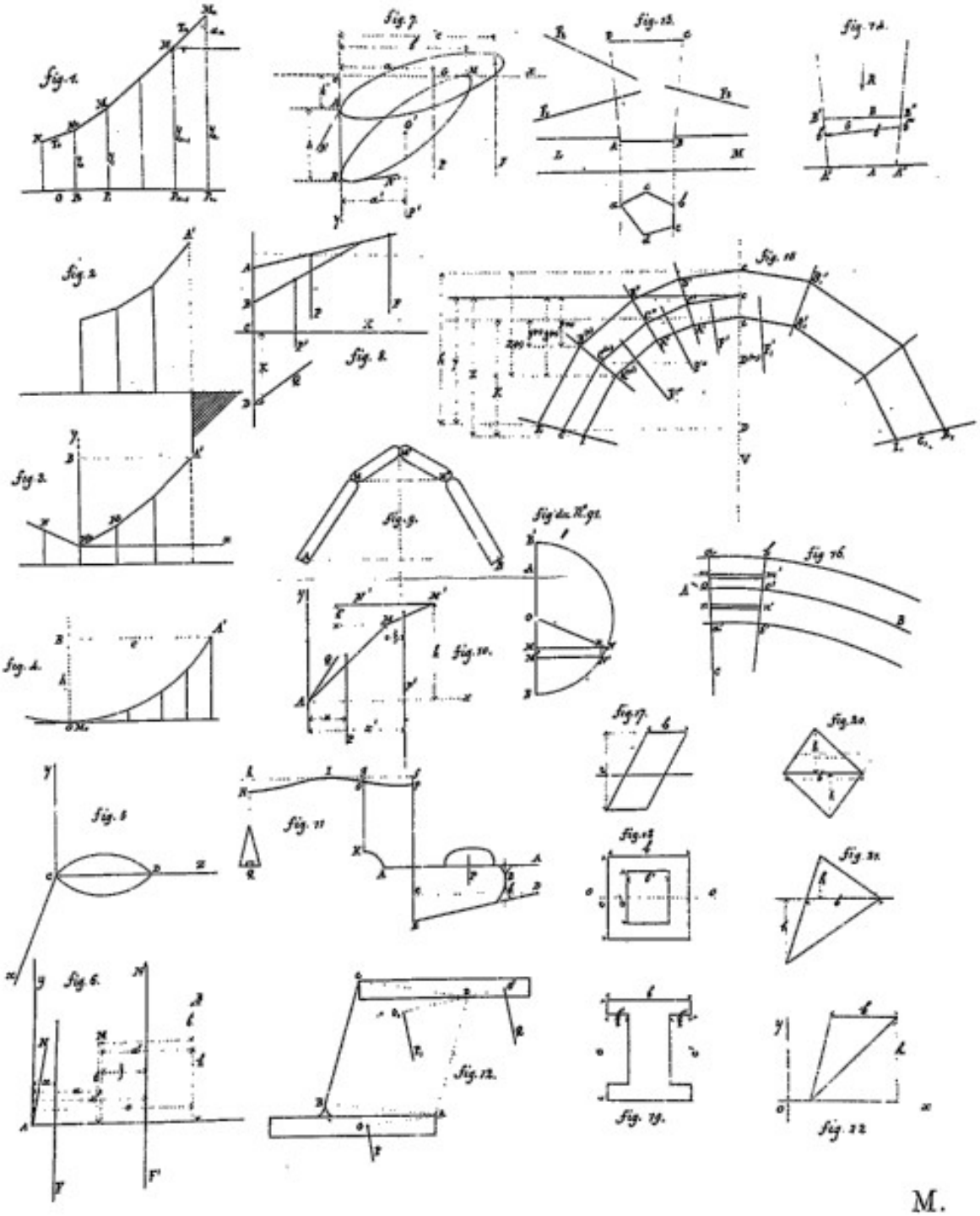
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fig. 22.
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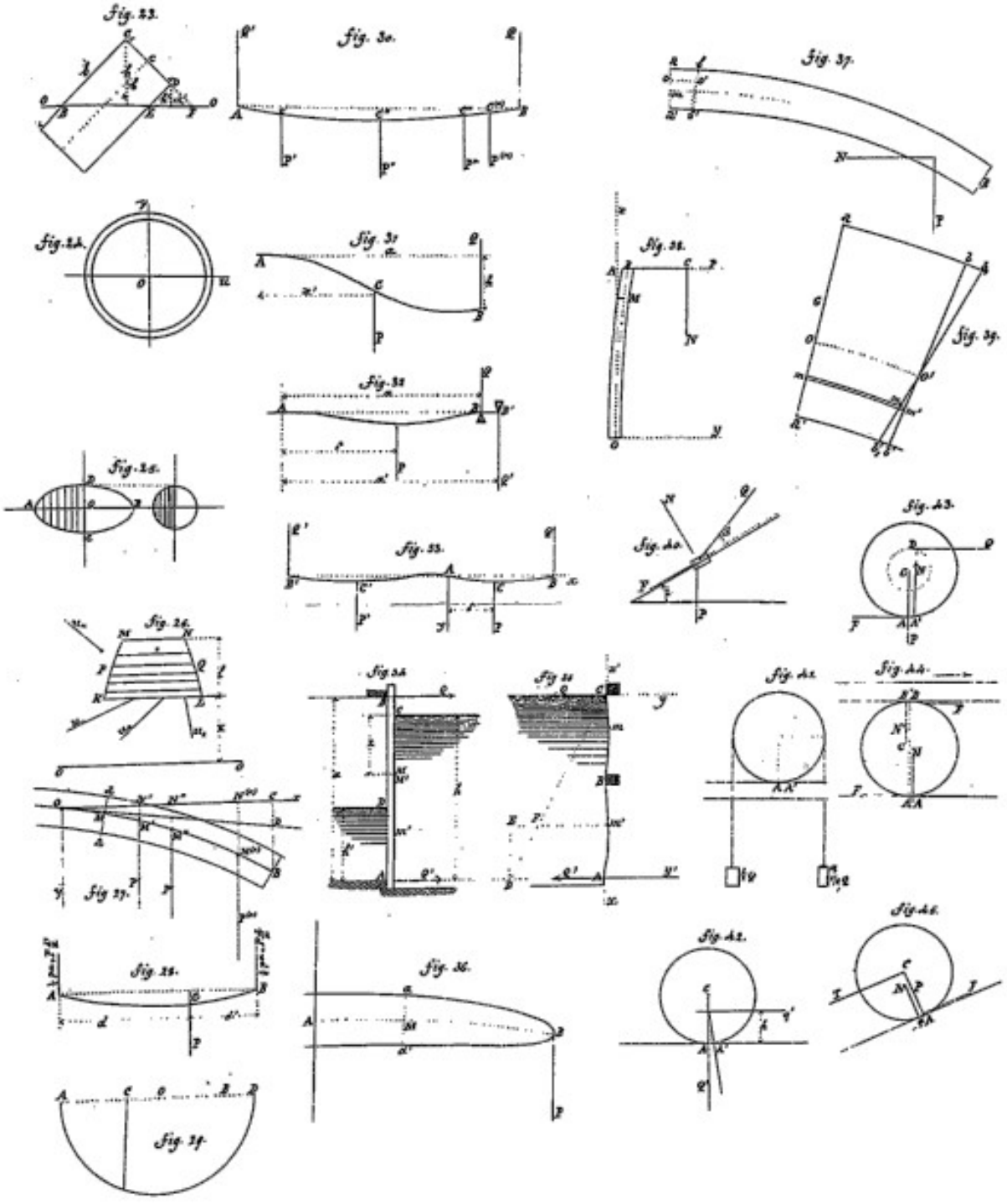
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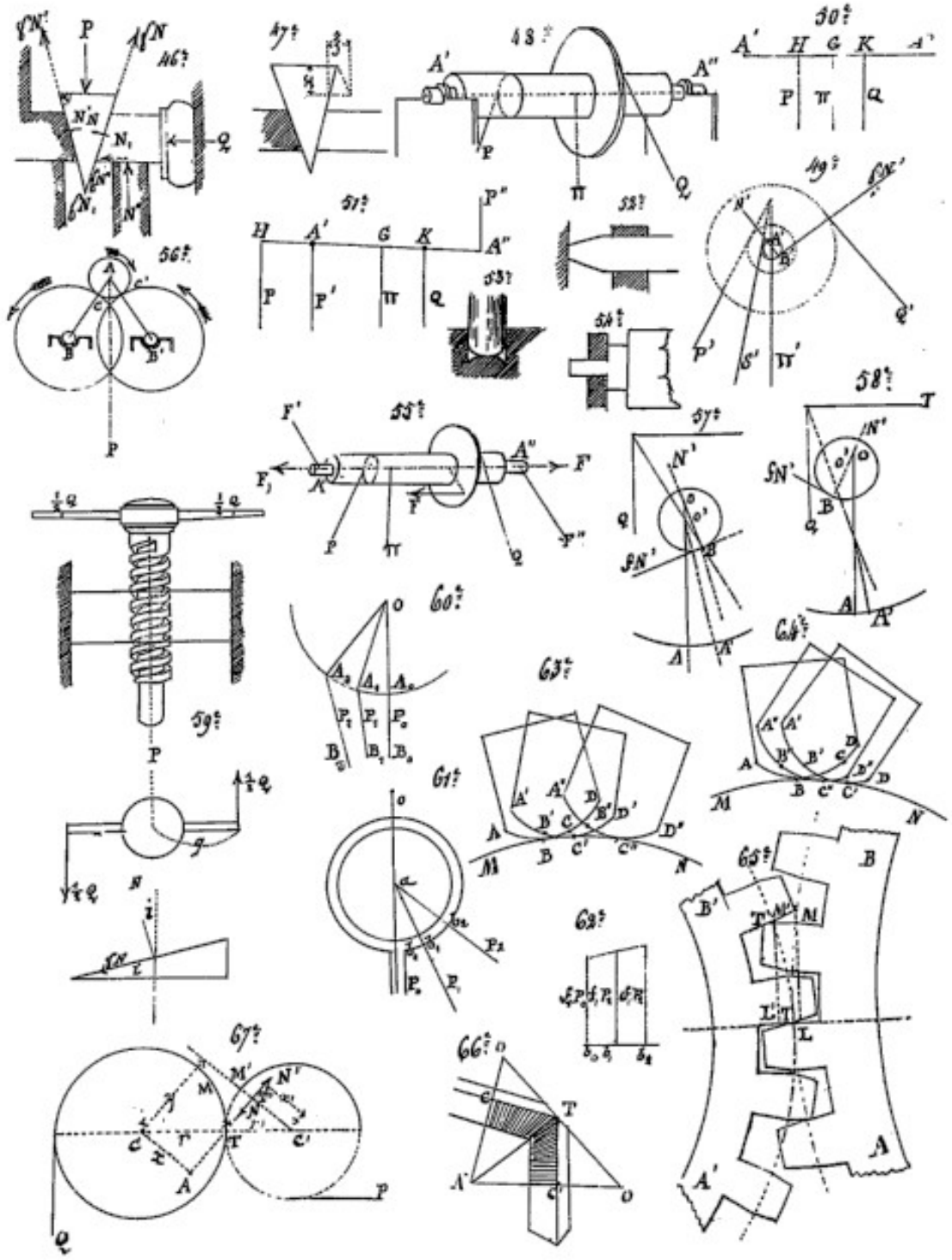

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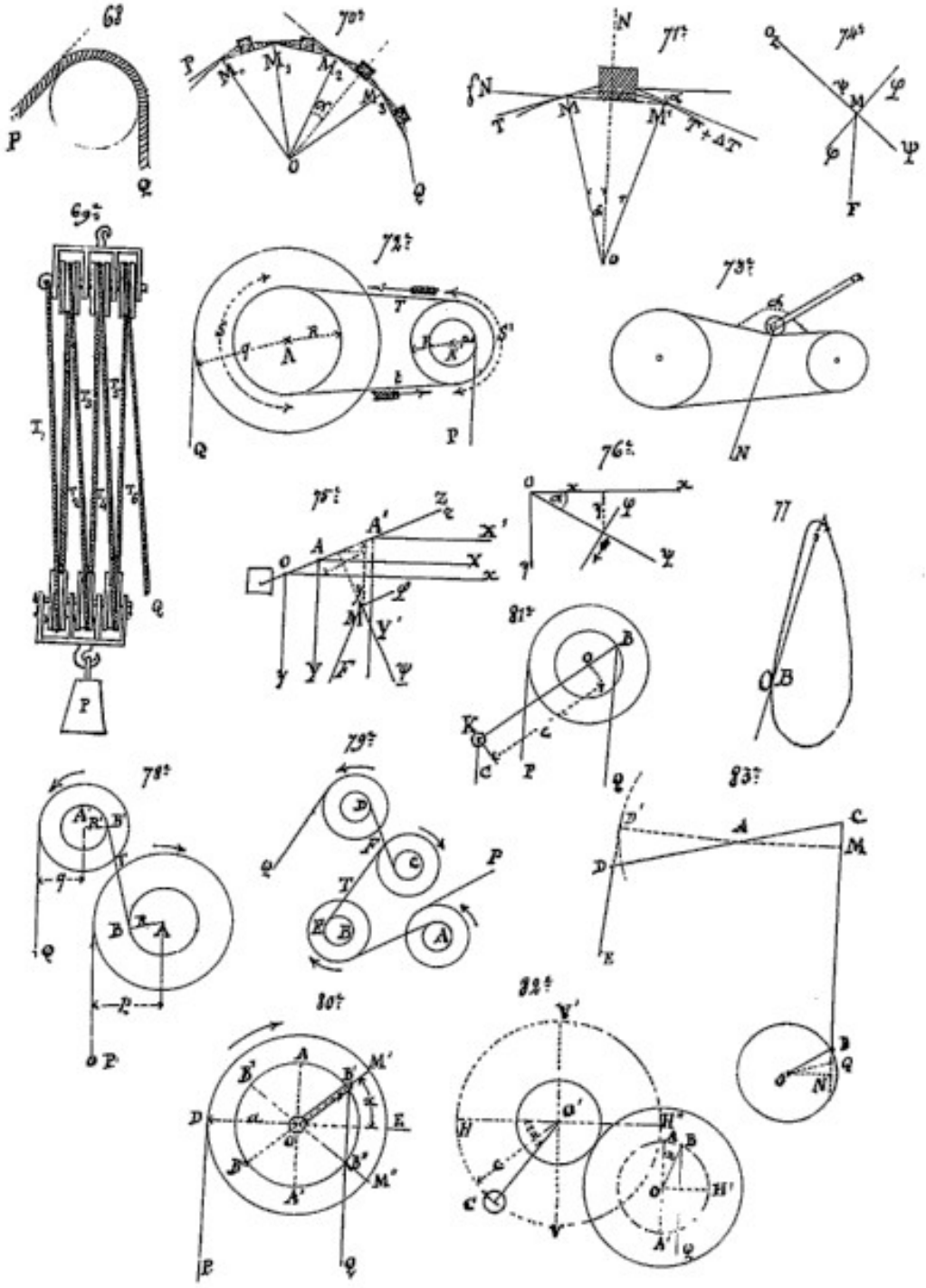

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